THE TEST FUNCTION CONJECTURE FOR PRO-p IWAHORI LOCAL MODELS OF GENERAL LINEAR GROUPS AND GENERAL SYMPLECTIC GROUPS

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Abstract. This paper is a continuation of [HLS]. Building on the enlarged local models of GL_n and GSp_{2g} at $\Gamma_1(p)$-level constructed in [HLS], and employing nearby cycles on these models, we prove that the function τ_μ^{ss} in the center of the $\Gamma_1(p)$-Hecke algebra, defined geometrically via the semisimple trace, coincides with the function z_μ^{ss} obtained from semisimple local Langlands parameters [FS] and the theory of the stable Bernstein center [Hai14]. This provides the first verification of the generalized test function conjecture at $\Gamma_1(p)$-level, valid for all cocharacters μ.

Contents

1. Introduction

Integral models control many arithmetic properties of Shimura varieties. Fix a prime p and a level structure K_p at p. It is important to construct an integral model over $\mathrm{Spec}(\mathcal{O}_F)$, where F is the completion of the reflex field at a finite place above p. When K_p is an *Iwahori* (or more generally a *parahoric*) subgroup in the sense of [BT], integral models are known for a large class of Shimura varieties: for PEL type, see [RZ96]; for abelian type, see [KP, KPZ]. One can also work in the p-adic setting: the existence of canonical p-adic integral models, conjectured by Pappas–Rapoport, has been established to varying degrees in [PR, D, DY, DHKZ]. Among the known cases, a prototypical example is the Siegel modular stack $\mathcal{A}_0$ of principally polarized abelian schemes of dimension g over $\mathrm{Spec}(\mathbb{Z}_p)$ with Iwahori level at p. This will be our motivating example for passing to deeper levels.

When K_p is a *pro-p Iwahori* subgroup (the pro-unipotent radical of an Iwahori subgroup), we say that the Shimura variety has $\Gamma_1(p)$-level structure at p. In this setting, far fewer results are known; see, for instance, work on certain unitary Shimura varieties [HRa, S] and on the Siegel case [HLS, S]. The Hilbert–Siegel case is studied in [Liu]. In [HLS], the $\Gamma_1(p)$ analogue $\mathcal{A}_1$ of $\mathcal{A}_0$ is constructed as a modular stack using Oort–Tate theory [OT]. Geometric properties of $\mathcal{A}_1$ have been investigated in [HLS, Mar].

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To address the bad reduction of $\mathcal{A}_0$, de Jong introduced a local model $\mathcal{M}_0$ in [dJ]. The scheme $\mathcal{M}_0$, defined in terms of linear algebra data, is projective over $\mathrm{Spec}(\mathbb{Z}_p)$ and is étale-locally isomorphic to $\mathcal{A}_0$; hence it captures the same singularities. Nearby cycles on $\mathcal{A}_0$ can be related to those on $\mathcal{M}_0$ via the *local model diagram*. For generalizations and applications in the parahoric case, see the survey [PRS].

Using the theory of determinant line bundles [KM, Knud] and their distinguished sections [HLS, Sec. 2], a local model $\mathcal{M}_1$ for $\mathcal{A}_1$ is constructed in [HLS] with properties parallel to those of $\mathcal{M}_0$. In particular, $\mathcal{M}_1$ is étale-locally isomorphic to $\mathcal{A}_1$, and there exists a local model diagram relating nearby cycles on $\mathcal{A}_1$ and $\mathcal{M}_1$. To relate *pushforwards* of nearby cycles, [HLS, Sec. 5] introduces rigidified local models by trivializing determinant line bundles and taking $(p-1)$-st roots of their distinguished sections (note that our notation differs from [HLS], where $(-)^+$ is used in place of $(-)^t$):

$$\mathcal{M}_1^t \longrightarrow \mathcal{M}_0^t \longrightarrow \mathcal{M}_0.$$

The morphism $\mathcal{M}_1^t \to \mathcal{M}_0^t$ is smooth-locally isomorphic to $\mathcal{A}_1 \to \mathcal{A}_0$, allowing us to compare, for a prime $\ell \neq p$,

$$\pi_* R\Psi_{\mathcal{A}_1}(\overline{\mathbb{Q}}_\ell) \qquad \text{and} \qquad \pi_* R\Psi_{\mathcal{M}_1^t}(\overline{\mathbb{Q}}_\ell),$$

for $\pi : \mathcal{A}_1 \to \mathcal{A}_0$ or $\pi : \mathcal{M}_1^t \to \mathcal{M}_0^t$. Thus, to study pushforwards of nearby cycles, it suffices to work with $\mathcal{M}_1^t \to \mathcal{M}_0^t \to \mathcal{M}_0$.

There are "enlarged" versions of $\mathcal{M}_0$ [HN, HLS] whose generic fibers carry interesting sheaves beyond $\overline{\mathbb{Q}}_\ell$. These enlarged local models $\mathcal{M}_{0,m}$ are indexed by $m = (m^+, m^-)$ with integers $m^- < m^+$. For $m = (1, 0)$ one recovers the usual local model $\mathcal{M}_0$. Variants exist for both $G = \mathrm{GL}_n$ and $G = \mathrm{GSp}_{2g}$, denoted $\mathcal{M}_{0,m,G}$. Unless stated otherwise, we focus on $G = \mathrm{GL}_n$ in this introduction, noting that the symplectic case has analogous statements.

The generic (resp. special) fiber of $\mathcal{M}_{0,m,G}$ embeds naturally into the affine Grassmannian $\mathrm{Gr}_{G,\mathbb{Q}_p}$ (resp. the affine flag variety $\mathrm{Fl}_{G,\mathbb{F}_p}$) via the lattice (resp. lattice-chain) description [Go, Zhu16]. Thus one may view $\mathcal{M}_{0,m,G}$ as a truncated deformation interpolating between $\mathrm{Fl}_{G,\mathbb{F}_p}$ and $\mathrm{Gr}_{G,\mathbb{Q}_p}$. A loop–group uniformization in the sense of [PZ] shows that these fiberwise embeddings can be made integrally [HLS, Sec. 9.5].

There are also enlarged versions of the rigidified models, yielding

$$\mathcal{M}_{1,m,G}^t \longrightarrow \mathcal{M}_{0,m,G}^t \longrightarrow \mathcal{M}_{0,m,G}.$$

Here $\mathcal{M}_{0,m,G}^t$ can be viewed as a truncated deformation from the enhanced affine flag variety $\mathrm{Fl}_{G,\mathbb{F}_p}^t$—which parametrizes lattice chains together with trivializations of consecutive cokernels—to a T-torsor (with T the diagonal torus of G) over $\mathrm{Gr}_{G,\mathbb{Q}_p}$. Using the embedding of the special fiber, I^+–equivariant nearby cycles on $\mathcal{M}_{0,m,G}^t$ produce elements of the $\Gamma_1(p)$ Hecke algebra $\mathcal{H}(G, I^+)$. One of the main results of this paper, made precise below, is that nearby cycles arising from natural constructions yield *central* elements of $\mathcal{H}(G, I^+)$, and that the action of these central elements on irreducible smooth representations of $G(\mathbb{F}_p(\!(t)\!))$ can be described explicitly. This was established for minuscule cocharacters in [HLS]; here we generalize it to arbitrary cocharacters.

It is worth emphasizing a key difference from the Iwahori case: at present there is no known loop–group uniformization for $\mathcal{M}_{0,m,G}^t$. The obstruction is that natural candidates produce generic fibers isomorphic to $\mathrm{Gr}_{G,\mathbb{Q}_p}$ itself, rather than to a T-torsor over $\mathrm{Gr}_{G,\mathbb{Q}_p}$, which is what the $\Gamma_1(p)$ rigidification demands.

1.1. **Formulation of the main result.** In this paper, we are going to prove a case of the *test function conjecture* in [Hai14].

Let G be either GL_n or GSp_{2g} over $\mathrm{Spec}(\mathbb{Z})$; unless stated otherwise, we focus on $G = \mathrm{GL}_n$. Let T denote the diagonal torus of G and B the "upper" Borel subgroup. For a scheme over the ring of p-adic integers $\mathbb{Z}_p$, write η (resp. s) for its generic (resp. special) fiber. Fix an arbitrary uniformizer $w_p \in \mathbb{Z}_p$. We use this w_p in the construction of $\mathcal{M}_{0,m,G}$ (Definition 4.1). When clear from context, we write G for $G(\mathbb{F}_p(\!(t)\!))$. For a precise setup, see §2.

Recall the rigidified local models

$$\mathcal{M}_{1,m,G}^t \xrightarrow{\pi} \mathcal{M}_{0,m,G}^t \longrightarrow \mathcal{M}_{0,m,G},$$

which are schemes over $\mathrm{Spec}(\mathbb{Z}_p)$, where $m = (m^+, m^-)$ is the truncation index as before. The scheme $\mathcal{M}^t_{0,m,G}$ is a T-torsor over $\mathcal{M}_{0,m,G}$ parametrizing consecutive determinant line bundles (Definition 6.1). The map $\mathcal{M}^t_{1,m,G} \to \mathcal{M}^t_{0,m,G}$ parametrizes $(p-1)$-st roots of distinguished sections of these determinant line bundles (Definition 8.1); it is a $T(\mathbb{F}_p)$-ramified cover and is finite étale on the generic fiber.

Fix a dominant cocharacter $\mu \in X_*(T)^+$. Let $\mathcal{A}_\mu$ (§8) be the usual normalized intersection complex on $\mathrm{Gr}_{G,\mathbb{Q}_p}$ attached to the Schubert stratification; this sheaf plays a central role in the geometric Satake equivalence [Lu, Gi, BD, MV, Ri14, RZ15, Zhu16]. For m sufficiently large (as we shall always assume), the embedding $\mathcal{M}_{0,m,G,\eta} \hookrightarrow \mathrm{Gr}_{G,\mathbb{Q}_p}$ (Lemma 4.4) allows us to view $\mathcal{A}_\mu$ as a perverse sheaf on $\mathcal{M}_{0,m,G}$, which we continue to denote by $\mathcal{A}_\mu$.

Write $\mathcal{B}^+_\mu$ for the pullback of $\mathcal{A}_\mu$ along the T-torsor $\mathcal{M}^t_{0,m,G} \to \mathcal{M}_{0,m,G}$. The analogue of $\mathcal{A}_\mu$ on $\mathcal{M}^t_{0,m,G}$ is then

$$\mathcal{A}^+_{\mu,m} \;:=\; \pi_*\pi^*(\mathcal{B}^+_\mu).$$

Define

$$\tau^{ss}_{\mu,m} : \mathcal{M}^t_{0,m,G}(\mathbb{F}_p) \to \overline{\mathbb{Q}}_\ell, \quad x \longmapsto (-1)^{d_\mu}\,\mathrm{Tr}^{ss}\Big(\Phi \mid (R\Psi_{\mathcal{M}^t_{0,m,G}}(\mathcal{A}^+_{\mu,m})_{\bar{x}})\Big),$$

where Tr^{ss} denotes the semisimple trace in the sense of Rapoport (see [HN, Sec. 3.1] and [PZ, Sec. 10.4]), $d_\mu = \dim(O_\mu)$ is the dimension of the Schubert stratum attached to μ, $R\Psi$ is the nearby cycle functor (§9.2), and $\Phi \in \mathcal{W}_{\mathbb{Q}_p}$ (the Weil group of $\mathbb{Q}_p$) is any geometric Frobenius element. Let $I \subset G(\mathbb{F}_p[\![t]\!])$ be the Iwahori subgroup corresponding to B, and let I^+ be its pro-unipotent radical. One checks that $R\Psi_{\mathcal{M}^t_{0,m,G}}(\mathcal{A}^+_{\mu,m})$ is I^+-equivariant (indeed, twisted I-equivariant in a suitable sense; see Lemma 10.1). Via the embedding $\mathcal{M}^t_{0,m,G,s} \hookrightarrow \mathrm{Fl}^t_{G,\mathbb{F}_p}$ (Lemma 6.2), we may view $\tau^{ss}_{\mu,m}$ as an element of the $\Gamma_1(p)$ Hecke algebra $\mathcal{H}\big(G(\mathbb{F}_p(\!(t)\!)), I^+\big)$ (see §9.1), and we keep the same notation for this function.

To describe the action of $\tau^{ss}_{\mu,m}$ on irreducible smooth representations of $G(\mathbb{F}_p(\!(t)\!))$ with coefficients in $\overline{\mathbb{Q}}_\ell$, we introduce a central function z^{ss}_μ. Let (r_μ, V_μ) be the μ-highest weight algebraic representation of ${}^LG = \widehat{G} \rtimes \mathcal{W}_{\mathbb{F}_p(\!(t)\!)}$ (with the dual group $\widehat{G}$ defined over $\overline{\mathbb{Q}}_\ell$ and $\mathcal{W}_{\mathbb{F}_p(\!(t)\!)}$ the Weil group of $\mathbb{F}_p(\!(t)\!)$). In our split setting, $\mathcal{W}_{\mathbb{F}_p(\!(t)\!)}$ acts trivially on V_μ, so V_μ is the usual highest-weight $\widehat{G}$-module; see [Hai14, Sec. 6.1]. There is an associated distribution $Z_\mu \in \mathfrak{Z}(G)$ (see [HLS, Sec. 13] and §9.3 for details), where $\mathfrak{Z}(G)$ denotes the Bernstein center of $G(\mathbb{F}_p(\!(t)\!))$. For an irreducible smooth representation π, let $\varphi_\pi : \mathcal{W}_{\mathbb{F}_p(\!(t)\!)} \to {}^LG$ be its semisimple Langlands parameter [FS], and set

$$Z_\mu(\pi) \;=\; \mathrm{Tr}\Big(r_\mu\big(\varphi_\pi(\Phi^\flat)\big) \;\Big|\; V_\mu^{\,r_\mu(\varphi_\pi(\mathcal{I}_{\mathbb{F}_p(\!(t)\!)}))}\Big),$$

where $\mathcal{I}_{\mathbb{F}_p(\!(t)\!)}$ is the inertia subgroup of $\mathbb{F}_p(\!(t)\!)$ and $\Phi^\flat \in \mathcal{W}_{\mathbb{F}_p(\!(t)\!)}$ is a geometric Frobenius element. This distribution yields an element in the center $\mathcal{Z}(G, I^+)$ of $\mathcal{H}(G, I^+)$. (Alternatively, z^{ss}_μ can be constructed using local class field theory and depth-zero type theory [BK, Hai12]—without invoking [FS]—but the construction above works uniformly at any level.) Up to a scalar, z^{ss}_μ is the *test function*: it is the expected function (cf. [Hai14, Conjecture 6.1.1]) inserted into the point-counting formula in the Langlands–Kottwitz method for computing the semisimple Lefschetz number.

The following theorem generalizes [HR21, Main Theorem] to the $\Gamma_1(p)$-level and can be regarded as a nearby-cycle construction of the test function.

Theorem 1.1 (Test function conjecture for $\Gamma_1(p)$ local models)**.** *For any m, the function $\tau^{ss}_{\mu,m}$ belongs to the center of $\mathcal{H}\big(G(\mathbb{F}_p(\!(t)\!)), I^+\big)$, and*

$$\tau^{ss}_{\mu,m} \;=\; |T(\mathbb{F}_p)|\,\alpha_t^{m^+}\big(z^{ss}_{\mu_{(m^+)}}\big) \quad \text{in } \mathcal{H}\big(G(\mathbb{F}_p(\!(t)\!)), I^+\big),$$

where $|T(\mathbb{F}_p)|$ is the cardinality of $T(\mathbb{F}_p)$, $\mu_{(m^+)} := \mu - (m^+, \ldots, m^+)$, and t is the parameter of $\mathbb{F}_p(\!(t)\!)$. The operator

$$\alpha_t : \mathcal{H}(G, I^+) \longrightarrow \mathcal{H}(G, I^+), \qquad \alpha_t(f)(z) = f\big((\mathrm{diag}(t, \ldots, t))^{-1} z\big)\,.$$

Moreover, when m^+ is divisible by $p-1$ we have $\alpha_t^{m^+}\big(z^{ss}_{\mu_{(m^+)}}\big) = z^{ss}_\mu$, and hence

$$\tau^{ss}_{\mu,m} \;=\; |T(\mathbb{F}_p)|\, z^{ss}_\mu\,.$$

The factor $|T(\mathbb{F}_p)|$ in the theorem comes from our normalization and may be ignored (see Example 1.4).

In the Iwahori case, the identity $\alpha_t^{m^+}(z^{ss}_{\mu_{(m^+)}}) = z^{ss}_\mu$ holds for every m^+. Consequently, $\tau^{ss}_{\mu,m} = z^{ss}_\mu$ for all m, in alignment with [HR21, Main Theorem]. This implies that $\tau^{ss}_{\mu,m}$ does not depend on m; this is immediate from the fact that the sheaf $\mathcal{A}_\mu$ itself is independent of m (the natural closed embeddings for different truncation indices yield a single uniform sheaf).

In contrast, at $\Gamma_1(p)$-level the operator α_t is genuinely needed. The sheaf $\mathcal{A}^+_{\mu,m}$ on $\mathcal{M}^t_{0,m,G}$ *does* depend on the truncation index m: although natural embeddings between the schemes $\mathcal{M}^t_{0,m,G}$ exist, there need not be an embedding $\mathcal{M}^t_{1,m,G} \hookrightarrow \mathcal{M}^t_{1,m',G}$ unless $m^+ - m'^+$ is divisible by $p-1$ (for GSp_{2g} this also forces $m^- - m'^-$ to be divisible by $p-1$). For a fixed μ, varying m can therefore produce different functions—indeed, at most $p-1$ of them (see Example 1.4 and §8.2).

The previous theorem extends to $\mathbb{F}_q$-points. Before explaining the extension, we outline the strategy of the proof of Theorem 1.1. The argument proceeds componentwise. Let $T(\mathbb{F}_p)^\vee$ be the set of characters $\chi : T(\mathbb{F}_p) \to \overline{\mathbb{Q}}_\ell^\times$. By [Hai12], any $z \in \mathcal{Z}(G, I^+)$ admits a decomposition (sum or average, depending on normalization)

$$z = \sum_{\chi \in T(\mathbb{F}_p)^\vee} z_\chi,$$

where $z_\chi \in \mathcal{Z}(G, I, \chi)$ lies in the center of the χ^{-1}–equivariant Hecke algebra $\mathcal{H}(G, I, \chi) \subset \mathcal{H}(G, I^+)$ of bi-χ^{-1}–equivariant functions (here we view χ as a character of $I \to I/I^+ = T(\mathbb{F}_p)$; see §9.1). Since $z^{ss}_\mu \in \mathcal{Z}(G, I^+)$, we write $z^{ss}_{\mu,\chi}$ for its χ–component (in the "average" sense).

Analogously (see [HLS, Sec. 14] and also Lemma 8.10 for a simpler description), there is a decomposition on the sheaf side

$$\mathcal{A}^+_{\mu,m} = \bigoplus_{\chi \in T(\mathbb{F}_p)^\vee} \mathcal{A}^+_{\mu,m,\chi},$$

where each $\mathcal{A}^+_{\mu,m,\chi}$ is χ–monodromic [Ve, LY, Gou, Zhu25, HLS], i.e. equivariant with respect to the rank–one multiplicative local system $\mathcal{F}_\chi$ [Lau]. The complexes

$$R\Psi_{\mathcal{M}^t_{0,m,G}}(\mathcal{A}^+_{\mu,m,\chi})$$

are again χ–monodromic and carry a *twisted* I–equivariance (distinct from the usual left I–action on the enhanced affine flag variety), which is used to deduce *bi*–χ^{-1}–equivariance. Taking semisimple traces yields functions

$$\tau^{ss}_{\mu,m,\chi} \in \mathcal{H}(G, I, \chi),$$

and hence

$$\tau^{ss}_{\mu,m} = \sum_{\chi \in T(\mathbb{F}_p)^\vee} \tau^{ss}_{\mu,m,\chi}.$$

A key input from [HLS] is that, for any m and χ, the function $\tau^{ss}_{\mu,m,\chi}$ is central in $\mathcal{H}(G, I, \chi)$, as expected.

The main step in the proof of Theorem 1.1 is to show that, when m^+ is divisible by $p-1$,

$$\tau^{ss}_{\mu,m,\chi} = z^{ss}_{\mu,\chi} \qquad \text{for all } \chi.$$

Following the philosophy of [HR21], we verify this by comparing constant terms:

$$c^G_T(\tau^{ss}_{\mu,m,\chi}) = c^G_T(z^{ss}_{\mu,\chi}),$$

where

$$c^G_T : \mathcal{Z}(G, I, \chi) \hookrightarrow \mathcal{Z}(T, T(\mathbb{F}_p[\![t]\!]), \chi)$$

is the (injective) constant–term morphism [Hai12, Sec. 5.4], given by

$$c^G_T(f)(m) = \delta_B^{1/2}(m) \cdot \int_{U(\mathbb{F}_p(\!(t)\!))} f(mu)\, du,$$

with U the unipotent radical of B.

Up to normalization, the constant–term morphism c^G_T can be realized geometrically via hyperbolic localization [Ri19, HR20, HR21] for a suitably *twisted* $\mathbb{G}_m$–action on $\mathcal{M}^t_{0,m,G}$ (the usual left action on $\mathrm{Fl}^t_{G,\mathbb{F}_p}$ has no fixed points); see §7 for definitions. The following theorem is an analogue of [HR21, Proposition 3.4, Proposition 4.7] at the $\Gamma_1(p)$–level. In what follows, $\mathcal{M}_{0,m,T}$ (resp. $\mathcal{M}_{0,m,B}$) denotes the analogue of $\mathcal{M}_{0,m,G}$ for T (resp. B); it is described in terms of lattices so that the corresponding T–torsor is defined naturally, and $(-)^t$ denotes the associated T–torsor. In the theorem below, the top rows of the diagrams arise functorially from the natural maps $T \leftarrow B \rightarrow G$. See §4–§7 for precise definitions.

Theorem 1.2. *Over* $\mathrm{Spec}(\mathbb{Q}_p)$*, there is a natural commutative diagram*

$$\begin{tikzcd}
\mathcal{M}^t_{0,m,T,\eta} \arrow[d] & \mathcal{M}^t_{0,m,B,\eta} \arrow[l] \arrow[r] \arrow[d] & \mathcal{M}^t_{0,m,G,\eta} \arrow[d, "="'] \\
\mathcal{M}^{t,0}_{0,m,G,\eta} & \mathcal{M}^{t,+}_{0,m,G,\eta} \arrow[l, "q^{t,+}"'] \arrow[r, "p^{t,+}"] & \mathcal{M}^t_{0,m,G,\eta}
\end{tikzcd}$$

in which all vertical maps are isomorphisms. Here $(-)^0$ *denotes the* $\mathbb{G}_m$*-fixed locus and* $(-)^+$ *the attractor.*

Similarly, over $\mathrm{Spec}(\mathbb{F}_p)$*, there is a natural commutative diagram*

$$\begin{tikzcd}
\mathcal{M}^t_{0,m,T,s} \arrow[d] & \mathcal{M}^t_{0,m,B,s} \arrow[l] \arrow[r] \arrow[d] & \mathcal{M}^t_{0,m,G,s} \arrow[d, "="'] \\
\mathcal{M}^{t,0}_{0,m,G,s} & \mathcal{M}^{t,+}_{0,m,G,s} \arrow[l, "q^{t,+}"'] \arrow[r, "p^{t,+}"] & \mathcal{M}^t_{0,m,G,s}
\end{tikzcd}$$

in which all vertical maps are isomorphisms.

Unlike the Iwahori case [HR21, Proposition 4.7], in the $\Gamma_1(p)$ setting the vertical maps are *isomorphisms* (rather than merely open-and-closed immersions), even on the special fiber. This makes the $\Gamma_1(p)$ case slightly better—though for applications the distinction is immaterial. Up to normalization, the constant–term morphism c^G_T can be realized (Lemma 10.9), via the sheaf-function dictionary, as

$$c^G_T \;\simeq\; (q^{t,+})_!\,(p^{t,+})^*,$$

and this functor commutes with nearby cycles on $\mathbb{G}_m$–equivariant sheaves [Ri19], such as $\mathcal{A}^+_{\mu,m,\chi}$. Consequently, all computations may be carried out on $\mathcal{M}^t_{0,m,T}$, whose reduced structure is comparatively simple (see Example 1.4).

Now we explain how to extend Theorem 1.1 to $\mathbb{F}_q$–points, where $q = p^r$ with $r \geq 1$. Let $I_r \subset G(\mathbb{F}_q[\![t]\!])$ be the corresponding Iwahori subgroup and I_r^+ its pro-unipotent radical. There are two natural constructions of nearby cycles in this setting. The need for two constructions stems from the fact that, although in the Iwahori case the double coset spaces agree

$$I\backslash G(\mathbb{F}_p(\!(t)\!))/I \;=\; I_r\backslash G(\mathbb{F}_q(\!(t)\!))/I_r,$$

this compatibility fails at $\Gamma_1(p)$ level:

$$I^+\backslash G(\mathbb{F}_p(\!(t)\!))/I^+ \;\neq\; I_r^+\backslash G(\mathbb{F}_q(\!(t)\!))/I_r^+.$$

First construction. For any $\chi \in T(\mathbb{F}_p)^\vee$, define

$$\tau^{ss}_{\mu,m,r,\chi} : \mathcal{M}^t_{0,m,G}(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell, \qquad x \longmapsto (-1)^{d_\mu}\,\mathrm{Tr}^{ss}\Big(\Phi^r \mid \big(R\Psi_{\mathcal{M}^t_{0,m,G}}(\mathcal{A}^+_{\mu,m,\chi})_{\bar{x}}\big)\Big).$$

Set $\tau^{ss}_{\mu,m,r} := \sum_{\chi\in T(\mathbb{F}_p)^\vee} \tau^{ss}_{\mu,m,r,\chi}$. Note that this sum only ranges over $T(\mathbb{F}_p)^\vee$, so from the point of view of $T(\mathbb{F}_q)^\vee$ (the set of characters of $T(\mathbb{F}_q)$) some components are missing.

Second construction. Define a $T(\mathbb{F}_q)$–ramified cover

$$\pi^r : \; \mathcal{M}^{t,r}_{1,m,G} \longrightarrow \mathcal{M}^t_{0,m,G}$$

as the space of $(q-1)$–st roots of the distinguished sections (for $r=1$ this is π). Set

$$\mathcal{A}^{+,r}_{\mu,m} := \pi^r_* \pi^{r*}(\mathcal{B}^+_\mu) \quad \text{on } \mathcal{M}^+_{0,m,\mathbb{Q}_p}.$$

After base change, we obtain on $\mathcal{M}^+_{0,m,\mathbb{Q}_q}$ a decomposition

$$\mathcal{A}^{+,r}_{\mu,m} = \bigoplus_{\chi' \in T(\mathbb{F}_q)^*} \mathcal{A}^{+,r}_{\mu,m,\chi'},$$

where each $\mathcal{A}^{+,r}_{\mu,m,\chi'}$ is χ'–monodromic. Define

$$\tau^{r,ss}_{\mu,m,\chi'} : \mathcal{M}^t_{0,m,G}(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell, \qquad x \longmapsto (-1)^{d_\mu} \operatorname{Tr}^{ss}\Big(\Phi^r \mid \big(R\Psi_{\mathcal{M}^t_{0,m,G}}(\mathcal{A}^{+,r}_{\mu,m,\chi'})_{\bar{x}}\big)\Big).$$

Then set $\tau^{r,ss}_{\mu,m} := \sum_{\chi' \in T(\mathbb{F}_q)^\vee} \tau^{r,ss}_{\mu,m,\chi'}$.

Spectral side. Replacing $\mathbb{F}_p(\!(t)\!)$ by $\mathbb{F}_q(\!(t)\!)$ in the construction of z^{ss}_μ yields a central function $z^{ss}_{\mu,r}$.

Both constructions produce central functions, and their relationship is summarized in the following theorem.

Theorem 1.3. *For any m, the function $\tau^{r,ss}_{\mu,m} \in \mathcal{H}\big(G(\mathbb{F}_q(\!(t)\!)), I^+_r\big)$ is central, and*

$$\tau^{r,ss}_{\mu,m} \;=\; |T(\mathbb{F}_q)|\, \alpha_t^{m^+}\big(z^{ss}_{\mu_{(m^+)},r}\big), \qquad \text{where } \mu_{(m^+)} := \mu - (m^+,\dots,m^+).$$

Moreover, for any $\chi \in T(\mathbb{F}_p)^*$ and $\chi' := \chi \circ N_{\mathbb{F}_q/\mathbb{F}_p}$ where $N_{\mathbb{F}_q/\mathbb{F}_p}$ is the usual norm map, we have

$$\tau^{ss}_{\mu,m,r,\chi} \;=\; \tau^{r,ss}_{\mu,m,\chi'},$$

and $\tau^{ss}_{\mu,m,r} \in \mathcal{H}\big(G(\mathbb{F}_q(\!(t)\!)), I^+_r\big)$ is central.

Moreover, if $r' \mid r$, then the sheaf $\mathcal{A}^{+,r'}_{\mu,m}$ is naturally a direct summand of $\mathcal{A}^{+,r}_{\mu,m}$, and the two sheaves induce the same function on $\mathbb{F}_{p^{r'}}$–points (whereas on $\mathbb{F}_{p^r}$–points they may differ, as noted above). If one wishes to recover $z^{ss}_{\mu,r}$ uniformly for all r, it is natural to consider the directed limit of the system $\{\mathcal{A}^{+,r}_{\mu,m}\}_r$ as r varies.

We end this subsection with an example that exhibits all the new phenomena at the $\Gamma_1(p)$-level, except for the extension to $\mathbb{F}_q$-points.

Example 1.4. In this example we treat the simplest case. Assume $G = T = \mathbb{G}_m$, take μ to be the trivial cocharacter, and work with $\mathbb{F}_p$-points. We have

$$I = \mathbb{F}_p[\![t]\!]^\times, \qquad I^+ = 1 + t\,\mathbb{F}_p[\![t]\!],$$

and

$$|T(\mathbb{F}_p)| = p-1.$$

The reader may ignore this factor of $p-1$, which arises from our normalization convention that z^{ss}_μ is the *average* rather than the sum of the $z^{ss}_{\mu,\chi}$'s. In this situation

$$z^{ss}_\mu = 1_{I^+} \qquad \text{and} \qquad z^{ss}_{\mu,\chi} = e_\chi,$$

where e_χ is the idempotent supported on I which, on I, is the function χ^{-1}, viewed as a character via the quotient

$$I/I^+ \;\simeq\; T(\mathbb{F}_p) = \mathbb{F}_p^\times.$$

First, consider the truncation index $m = (0,-1)$. Then the reduced scheme $(\mathcal{M}_{0,m,G})_{\mathrm{red}}$ (see Definition 4.8 and Lemma 10.6 for details) consists of two disjoint copies of $\operatorname{Spec}(\mathbb{Z}_p)$. One component corresponds to the middle term in

$$\mathbb{Z}_p[t] \subset \mathbb{Z}_p[t] \subset t^{-1}\mathbb{Z}_p[t],$$

and the other to the middle term in

$$\mathbb{Z}_p[t] \subset t^{-1}\mathbb{Z}_p[t] \subset t^{-1}\mathbb{Z}_p[t].$$

We denote these components by $\operatorname{Spec}(\mathbb{Z}_p)_0$ and $\operatorname{Spec}(\mathbb{Z}_p)_{-1}$, respectively. In this case $\mathcal{A}_\mu$ is simply the rank-one constant sheaf $\overline{\mathbb{Q}}_\ell$ on $\operatorname{Spec}(\mathbb{Q}_p)_0$, the generic fiber of $\operatorname{Spec}(\mathbb{Z}_p)_0$.

The reduced structure $(\mathcal{M}^t_{0,m,G})_{\mathrm{red}}$ of the $T_{\mathbb{Z}_p}$-torsor $\mathcal{M}^t_{0,m,G}$ is the disjoint union of two copies of $T_{\mathbb{Z}_p}$, one lying over each copy of $\mathrm{Spec}(\mathbb{Z}_p)$. We denote them by $T_{0,\mathbb{Z}_p}$ and $T_{-1,\mathbb{Z}_p}$. We trivialize the T-torsor so that, via the embedding (Lemma 6.2)

$$\mathcal{M}^t_{0,m,G,s} \hookrightarrow \mathrm{Fl}^t_{G,\mathbb{F}_p},$$

the points $1 \in T_{0,\mathbb{Z}_p}(\mathbb{F}_p)$ and $1 \in T_{-1,\mathbb{Z}_p}(\mathbb{F}_p)$ map to

$$1I^+ \in \mathrm{Fl}^t_{G,\mathbb{F}_p}(\mathbb{F}_p) = \mathbb{F}_p(\!(t)\!)^\times/(1+t\mathbb{F}_p[\![t]\!]) \quad \text{and} \quad t^{-1}I^+ \in \mathrm{Fl}^t_{G,\mathbb{F}_p}(\mathbb{F}_p),$$

respectively.

For any character $\chi : T(\mathbb{F}_p) = \mathbb{F}_p^\times \to \overline{\mathbb{Q}}_\ell^\times$, the sheaf $\mathcal{A}^+_{\mu,m,\chi}$ equals

$$\mathcal{F}_{\chi,\mathbb{Q}_p} \quad \text{on } T_{0,\mathbb{Q}_p},$$

the generic fiber of $T_{0,\mathbb{Z}_p}$, where $\mathcal{F}_{\chi,\mathbb{Q}_p}$ is the rank-one local system defined by the connected finite étale cover

$$T_{\mathbb{Q}_p} \xrightarrow{a \mapsto a^{p-1}} T_{\mathbb{Q}_p}$$

with Galois group $T(\mathbb{F}_p)$ and character χ. This is because over $T_{0,\mathbb{Q}_p}$, the morphism $\mathcal{M}^t_{1,m,G} \to \mathcal{M}^t_{0,m,G}$ can be identified with $T_{\mathbb{Q}_p} \xrightarrow{a \mapsto a^{p-1}} T_{\mathbb{Q}_p}$ (see Lemma 10.7 for details). It is also known ([HLS, Lemma 14.1.1]) that

$$R\Psi_{T_{\mathbb{Z}_p}}(\mathcal{F}_{\chi,\mathbb{Q}_p}) = \mathcal{F}_{\chi,\mathbb{F}_p},$$

the $\mathbb{F}_p$-analogue defined similarly. Taking the (usual or semisimple) Frobenius trace on $\mathcal{F}_{\chi,\mathbb{F}_p}$ yields the function e_χ ([HLS, Lemma 14.4.1]). Hence $\tau^{ss}_{\mu,m,\chi}$ is the function e_χ, and therefore

$$\tau^{ss}_{\mu,m} = (p-1)\,1_{I^+} = (p-1)\,z^{ss}_\mu.$$

Moreover, if one works with the truncation index $n = (n^+, n^-)$ where $n^+ \geq 0$ and n^+ is divisible by $p-1$ (with the same μ), then over $T_{0,\mathbb{Q}_p}$ the morphism $\mathcal{M}^t_{1,m,G} \to \mathcal{M}^t_{0,m,G}$ can be identified with the composite

$$T_{\mathbb{Q}_p} \xrightarrow{a \mapsto a^{p-1}} T_{\mathbb{Q}_p} \xrightarrow{a \mapsto a w_p^{-n^+}} T_{\mathbb{Q}_p}$$

(again see Lemma 10.7). One checks that we still obtain $\mathcal{F}_{\chi,\mathbb{Q}_p}$ on $T_{0,\mathbb{Q}_p}$ and thus the same equality.

Now consider the truncation index $m' = (1,0)$. By a similar analysis, $(\mathcal{M}^t_{0,m',G})_{\mathrm{red}}$ is the disjoint union of two copies of $T_{\mathbb{Z}_p}$, which we denote by $T_{0,\mathbb{Z}_p}$ and $T_{1,\mathbb{Z}_p}$. For any character $\chi : T(\mathbb{F}_p) \to \overline{\mathbb{Q}}_\ell^\times$, the sheaf $\mathcal{A}^+_{\mu,m',\chi}$ is $\mathcal{F}_{\chi,\mathbb{Q}_p} \otimes \mathcal{K}_\chi$ on $T_{0,\mathbb{Q}_p}$, where $\mathcal{K}_\chi$ is the rank-one local system on $\mathrm{Spec}(\mathbb{Q}_p)$ obtained by restricting $\mathcal{F}_{\chi,\mathbb{Q}_p}$ to the point $w_p \in T(\mathbb{Q}_p)$. This is because now over $T_{0,\mathbb{Q}_p}$, the morphism $\mathcal{M}^t_{1,m',G} \to \mathcal{M}^t_{0,m',G}$ can be identified with the composite

$$T_{\mathbb{Q}_p} \xrightarrow{a \mapsto a^{p-1}} T_{\mathbb{Q}_p} \xrightarrow{a \mapsto a w_p^{-1}} T_{\mathbb{Q}_p}.$$

Concretely, $\mathcal{K}_\chi$ corresponds to the character

$$\beta : \ \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \ \longrightarrow \ \mathrm{Gal}\left(\mathbb{Q}_p\left(w_p^{\frac{1}{p-1}}\right)/\mathbb{Q}_p\right) \cong \mathbb{F}_p^\times \xrightarrow{\chi^{-1}} \overline{\mathbb{Q}}_\ell^\times.$$

The inertia group $\mathcal{I}_{\mathbb{Q}_p} \subset \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts trivially on $\mathcal{K}_\chi$ if and only if χ is trivial; in that case $\mathcal{K}_\chi = \overline{\mathbb{Q}}_\ell$ on $\mathrm{Spec}(\mathbb{Q}_p)$. Consequently, the semisimple trace

$$\mathrm{Tr}^{ss}\big(\Phi \mid R\Psi_{\mathbb{Z}_p}(\mathcal{K}_\chi)_{\bar{s}}\big) = \begin{cases} 0, & \chi \neq 1, \\ 1, & \chi = 1. \end{cases}$$

Putting this together, we obtain

$$\tau^{ss}_{\mu,m',\chi} = \begin{cases} 0, & \chi \neq 1, \\ 1_I, & \chi = 1, \end{cases} \qquad \text{hence} \qquad \tau^{ss}_{\mu,m'} = 1_I \neq (p-1)1_{I^+} = (p-1)z^{ss}_\mu.$$

As one can see, for the same μ, different truncation indices m and m' give different functions.

For the cocharacter $\mu' : x \mapsto x^{-1}$ and the truncation index m', a similar computation shows that

$$(p-1)\,z^{ss}_{\mu'} = \tau^{ss}_{\mu',m'} = 1_{It^{-1}I},$$

and therefore

$$\tau^{ss}_{\mu,m} = 1_I = (p-1)\,\alpha_t\big(z^{ss}_{\mu'}\big)\,.$$

In particular, when χ is trivial,

$$\tau^{ss}_{\mu,m,\chi} = \alpha_t\big(z^{ss}_{\mu',\chi}\big) = \alpha_t(1_{It^{-1}I}) = 1_I = z^{ss}_{\mu,\chi}.$$

Thus, although α_t is unnecessary in the Iwahori case, it plays an essential role at the $\Gamma_1(p)$-level.

1.2. **Overview.** This paper is a continuation of [HLS], but it is intended to be self-contained. The essential input we take from [HLS] is the centrality of each χ-monodromic component. Apart from this, most results from [HLS] that we use are revisited and, where convenient, reproved here—sometimes with slight variations.

In §2, we set up the notation used throughout the paper. From this point on, we assume $G = \mathrm{GL}_n$. In §3, we review lattice descriptions of the affine Grassmannian and the (enhanced) affine flag variety. Sections §4 and §5 construct local models at Iwahori level and establish results on hyperbolic localization for these schemes. Sections §6 and §7 construct local models at $\Gamma_1(p)$-level and analyze hyperbolic localization there. In §8 and §9, we construct central elements in the $\Gamma_1(p)$ Hecke algebra and its variants. In §10, we prove that the central elements constructed by different methods coincide. Section §11 extends the preceding results to a general finite field $\mathbb{F}_q$. Finally, in §12, we introduce the additional ingredients needed to treat $G = \mathrm{GSp}_{2g}$ and indicate the corresponding generalizations.

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2. Notation

In this section, we fix the notation and conventions that will be used throughout the paper.

Fix $\ell \neq p$ two prime numbers throughout this paper. Let G be either GL_n or GSp_{2g} over $\mathrm{Spec}(\mathbb{Z})$. For GL_n, we view G as the automorphism group scheme of $\mathbb{Z}^n$, where we denote by e_i the element whose i-th coordinate is 1 and all other coordinates are 0. For GSp_{2g}, we view G as the automorphism group scheme of $\mathbb{Z}^{2g}$ preserving the standard symplectic form up to a unit, where the standard symplectic form $\langle \cdot, \cdot \rangle$ on $\mathbb{Z}^{2g}$ is given by the matrix

$$\begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix},$$

and J denotes the antidiagonal identity matrix of size $g \times g$.

Let $B/\mathrm{Spec}(\mathbb{Z})$ be the "upper" Borel subgroup and $T/\mathrm{Spec}(\mathbb{Z})$ the diagonal torus of G. When we work over a valuation field F (e.g. $\mathbb{Q}_p$, $\mathbb{F}_p(\!(t)\!)$, or $\mathbb{Q}_p(\!(t)\!)$) with valuation ring $\mathcal{O}_F$, where the Bruhat–Tits theory [BT] applies, we use the following conventions: by a *hyperspecial subgroup*, we always mean $G(\mathcal{O}_F)$; by an *Iwahori subgroup*, we mean the preimage of $B(F/\mathcal{O}_F) \subset G(F/\mathcal{O}_F)$ under the reduction map $G(F) \to G(F/\mathcal{O}_F)$; and by the $\Gamma_1(p)$ (or *pro-p*) Iwahori subgroup, we mean the preimage of $U(F/\mathcal{O}_F) \subset G(F/\mathcal{O}_F)$ under the same reduction map, where $U/\mathrm{Spec}(\mathbb{Z}) \subset B$ is the unipotent radical of B.

Occasionally, by "hyperspecial" (resp. "Iwahori") subgroup we also refer to the affine smooth connected $\mathcal{O}_F$-model of G_F whose set of $\mathcal{O}_F$-valued points is the hyperspecial (resp. Iwahori) subgroup defined above; this will be clear from the context.

When F is a local field, we fix a separable closure $\overline{F}$ of F and denote by W_F the Weil group of F, viewed as a subgroup of the Galois group $\mathrm{Gal}(\overline{F}/F)$. The Artin map

$$\mathrm{Art}_F^{-1} \colon W_F \longrightarrow F^\times$$

is normalized so that geometric Frobenius elements are sent to uniformizers. The inertia group of F is denoted by $J_F \subset \mathrm{Gal}(\overline{F}/F)$

Over a general field k, the *affine Grassmannian* $\mathrm{Gr}_{G,k}$ (resp. *affine flag variety* $\mathrm{Fl}_{G,k}$) attached to G is defined to be the ind-scheme representing the fpqc-sheafification of the functor

$$\{\, k\text{-algebra } R \longmapsto G(R(\!(t)\!))/G(R[\![t]\!])\,\} \quad (\text{resp. } \{\, k\text{-algebra } R \longmapsto G(R(\!(t)\!))/\mathcal{I}(R[\![t]\!])\,\}).$$

where $\mathcal{I}(R[\![t]\!]) \subset G(R[\![t]\!])$ is the preimage of $B(R) \subset G(R)$ under the reduction map $G(R[\![t]\!]) \to G(R)$).

When the base field k is clear from the context (in particular, when we work over the generic point $\mathrm{Spec}(\mathbb{Q}_p)$ or the special point $\mathrm{Spec}(\mathbb{F}_p)$ of $\mathrm{Spec}(\mathbb{Z}_p)$), we simply write Gr_G (resp. Fl_G) for $\mathrm{Gr}_{G,k}$ (resp. $\mathrm{Fl}_{G,k}$). The same convention applies to T and B: we set $\mathrm{Gr}_T = \mathrm{Fl}_T$ and $\mathrm{Gr}_B = \mathrm{Fl}_B$, defined as the fpqc-sheafification of the functor

$$\{\, k\text{-algebra } R \longmapsto T(R(\!(t)\!))/T(R[\![t]\!])\,\} \quad (\text{resp. } \{\, k\text{-algebra } R \longmapsto B(R(\!(t)\!))/B(R[\![t]\!])\,\}).$$

We fix a uniformizer $w_p \in \mathbb{Z}_p$ throughout this paper. For a scheme X defined over $\mathrm{Spec}(\mathbb{Z}_p)$, we denote by $X_\eta/\mathrm{Spec}(\mathbb{Q}_p)$ its generic fiber, by $X_s/\mathrm{Spec}(\mathbb{F}_p)$ its special fiber, by $X_{\bar\eta}/\mathrm{Spec}(\overline{\mathbb{Q}}_p)$ its geometric generic fiber (using the separable closure fixed above), and by $X_{\bar s}/\mathrm{Spec}(\overline{\mathbb{F}}_p)$ its geometric special fiber.

For each ring $\mathbf{k} \in \{\mathbb{Z}_p, \overline{\mathbb{Z}}_p, \mathbb{Q}_p, \overline{\mathbb{Q}}_p, \mathbb{F}_p, \overline{\mathbb{F}}_p\}$, the map $(p-1) : T_{\mathbf{k}} \longrightarrow T_{\mathbf{k}},\ t \mapsto t^{p-1}$ is a connected finite étale cover with covering group the $(p-1)$-st roots of the identity of $T(\mathbb{Z}_p)$, which we identify with $T(\mathbb{F}_p)$ via the Teichmüller lifting throughout the paper. We denote by $T(\mathbb{F}_p)^\vee$ the set of characters of $T(\mathbb{F}_p)$ valued in $\overline{\mathbb{Q}}_\ell^\times$.

3. Affine grassmannian and (enhanced) affine flag variety

Starting from this section, we assume that $G = \mathrm{GL}_n$, and we only deal with GSp_{2g} in the last section.

Let R be any ring. By an $R[\![t]\!]$-lattice in $R(\!(t)\!)^n$, we mean one of the following equivalent notions (see, e.g., [Go, Definition 3.1], [Zhu16, Definition 1.1.1], and [Hai25+]).

Definition 3.1. An $R[\![t]\!]$-lattice $\mathfrak{L} \subset R(\!(t)\!)^n$ is an $R[\![t]\!]$-submodule such that $t^N R[\![t]\!]^n \subseteq \mathfrak{L} \subseteq t^{-N} R[\![t]\!]^n$ for some integer $N > 0$, and that satisfies the following equivalent conditions:

(i) the R-module $t^{-N} R[\![t]\!]^n/\mathfrak{L}$ is R-projective;
(ii) the $R[\![t]\!]$-module $\mathfrak{L}$ is $R[\![t]\!]$-projective.

This definition extends to any scheme S by gluing along affine open subsets $U \subset S$; in this case we call such an object an $\mathcal{O}_S[\![t]\!]$-lattice.

Remark 3.2. For $N' > N > 0$ in Definition 3.1, using the natural sequence

$$t^{-N} R[\![t]\!]^n/\mathfrak{L} \longrightarrow t^{-N'} R[\![t]\!]^n/\mathfrak{L} \longrightarrow t^{-N'} R[\![t]\!]^n/t^{-N} R[\![t]\!]^n,$$

whose last term is R-projective, we see that the R-module $t^{-N} R[\![t]\!]^n/\mathfrak{L}$ is projective if and only if $t^{-N'} R[\![t]\!]^n/\mathfrak{L}$ is projective. Note that in Definition 3.1 we may equivalently replace N by two integers $m^+ > m^-$ such that $t^{m^+} R[\![t]\!]^n \subseteq \mathfrak{L} \subseteq t^{m^-} R[\![t]\!]^n$.

There is a standard lattice

$$\Lambda_{0,R} := R[\![t]\!]^n \subset R(\!(t)\!)^n.$$

Definition 3.1 provides an equivalent description of $\mathrm{Gr}_{G,k}$ as a functor.

Lemma 3.3. *Let R be any k-algebra. Then $\mathrm{Gr}_{G,k}(R)$ identifies functorially with the set of $R[\![t]\!]$-lattices in $R(\!(t)\!)^n$ via the map $g \in G(R(\!(t)\!)) \mapsto g\Lambda_{0,R}$.*

Proof. See [Zhu16, Definition 1.1.2, (1.2.1), and Proposition 1.3.6]. Note that GL_n-torsors can be naturally identified with rank-n vector bundles ([Zhu16, §0.3.3]). The main point is that any $R[\![t]\!]$-lattice $\mathfrak{L} \subset R(\!(t)\!)^n$ is étale locally on $\mathrm{Spec}(R)$ free ([Zhu16, Lemma 1.3.7]), and even Zariski locally on $\mathrm{Spec}(R)$ free ([Hai25+, Lemma 2.3.8]). Via the map in the lemma, the group $G(R(\!(t)\!))$ acts transitively on the set of free $R[\![t]\!]$-lattices in $R(\!(t)\!)^n$, with stabilizer $G(R[\![t]\!])$. □

A *t-periodic complete $R[\![t]\!]$-lattice chain* $\mathfrak{L}_\bullet := (\cdots \subset \mathfrak{L}_{-1} \subset \mathfrak{L}_0 \subset \mathfrak{L}_1 \subset \cdots)$ is a sequence of $R[\![t]\!]$-lattices such that $\mathfrak{L}_i/\mathfrak{L}_{i-1}$ is locally free of rank 1 and $\mathfrak{L}_{i+n} = t^{-1}\mathfrak{L}_i$ for all $i \in \mathbb{Z}$. There is a standard t-periodic complete $R[\![t]\!]$-lattice chain

$$\Lambda_{i,R} := \left(t^{-1}R[\![t]\!]\right)^i \oplus R[\![t]\!]^{2g-i}.$$

The notion of t-periodic complete $R[\![t]\!]$-lattice chains gives an equivalent description of $\mathrm{Fl}_{G,k}$ as a functor.

Lemma 3.4. *Let R be any k-algebra. Then $\mathrm{Fl}_{G,k}(R)$ identifies functorially with the set of t-periodic complete $R[\![t]\!]$-lattice chains in $R(\!(t)\!)^n$ via the map $g \in G(R(\!(t)\!)) \mapsto g(\Lambda_{\bullet,R})$.*

Proof. A t-periodic complete $R[\![t]\!]$-lattice chain $(\cdots \subset \mathfrak{L}_{-1} \subset \mathfrak{L}_0 \subset \mathfrak{L}_1 \subset \cdots)$ is determined by $\mathfrak{L}_0 \subset \mathfrak{L}_1 \subset \cdots \subset \mathfrak{L}_n = t^{-1}\mathfrak{L}_0$. Étale locally on $\mathrm{Spec}(R)$, we may assume that all $\mathfrak{L}_i$ and $\mathfrak{L}_i/\mathfrak{L}_{i-1}$ are free. We then obtain a full flag

$$\mathfrak{L}_0 \subset \mathfrak{L}_1/\mathfrak{L}_0 \subset \cdots \subset \mathfrak{L}_n/\mathfrak{L}_0 = t^{-1}\mathfrak{L}_0/\mathfrak{L}_0$$

in $t^{-1}\mathfrak{L}_0/\mathfrak{L}_0$. Since $G(R)$ acts transitively on the set of full flags in R^n with stabilizer $B(R)$, Lemma 3.3 implies the claim. □

Remark 3.5. In [HR21, §4], the affine flag variety $\mathrm{Fl}_{G,k}$ is defined as the fpqc-sheafification of the functor $\{\, k\text{-algebra } R \mapsto G(R(\!(t)\!))/\mathcal{I}(R[\![t]\!]) \,\}$, where $\mathcal{I}/\mathrm{Spec}(k[\![t]\!])$ is the Iwahori group scheme corresponding to B. This agrees with our definition, since for any k-algebra R, $\mathcal{I}(R[\![t]\!]) \subset G(R[\![t]\!])$ is the preimage of $B(R) \subset G(R)$ under the reduction map $G(R[\![t]\!]) \to G(R)$ (for k itself, this follows from Bruhat–Tits theory [BT]). For $G = \mathrm{GL}_n$, this is proved in [Hai05, §3.2]; the key point is that the automorphism group scheme $\underline{\mathrm{Aut}}(\Lambda_{\bullet,k[\![t]\!]})$ is smooth over $\mathrm{Spec}(k[\![t]\!])$, and hence equals the Iwahori group scheme. For $G = \mathrm{GSp}_{2g}$, smoothness is proved in [HLS, Proposition 6.2.4] (only the special fiber is needed).

Since we will later work with objects at the $\Gamma_1(p)$-level, we introduce the following definition of the *enhanced affine flag variety*.

Definition 3.6. Let $\mathrm{Fl}^t_{G,k}$ denote the T-torsor over $\mathrm{Fl}_{G,k}$ such that, for any k-algebra R, its R-valued points parametrize isomorphisms

$$\phi_i \colon \mathfrak{L}_i/\mathfrak{L}_{i-1} \xrightarrow{\cong} R, \quad 1 \le i \le n.$$

There is a standard R-valued point

$$\Lambda^t_{\bullet,R} \in \mathrm{Fl}^t_{G,k}(R)$$

whose underlying lattice chain is the standard chain $(\Lambda_{\bullet,R})$, and whose trivializations are given by

$$\phi_i \colon t^{-1}R[\![t]\!]/R[\![t]\!] \xrightarrow{\times t} R[\![t]\!]/tR[\![t]\!] \xrightarrow{t\mapsto 0} R.$$

Lemma 3.7. *Let R be any k-algebra. Then $\mathrm{Fl}_{G,k}$ identifies functorially with the R-valued points of the fpqc-sheafification of the functor $\{\, k\text{-algebra } R \mapsto G(R(\!(t)\!))/U(R[\![t]\!]) \,\}$, where $U(R[\![t]\!])$ is the preimage of $U(R) \subset G(R)$ under the reduction map $G(R[\![t]\!]) \to G(R)$. This identification is given by $g \in G(R(\!(t)\!)) \mapsto g(\Lambda^t_{\bullet,R})$.*

Proof. Consider the set of full flags in R^n equipped with trivializations of the consecutive quotients. Then $G(R)$ acts transitively on this set with stabilizer $U(R)$. The rest of the argument follows as in the proof of Lemma 3.4. □

4. Truncated deformations for the Iwahori level

As mentioned in §3, unless otherwise specified, $G = \mathrm{GL}_n$.

As in [HLS, Sec. 6.1], we define

$$\mathbb{V}_i[t] := (t + w_p)^{-1}\mathbb{Z}_p[t]^i \oplus \mathbb{Z}_p[t]^{n-i}$$

as a $\mathbb{Z}_p[t]$-submodule of

$$\mathbb{V} := \mathbb{Z}_p[t, t^{-1}, (t + w_p)^{-1}]^n = \mathbb{Z}^n \otimes \mathbb{Z}_p[t, t^{-1}, (t + w_p)^{-1}].$$

We recall the definition of the truncated deformations for the Iwahori level in [HN, Definition 2] and [HLS, Definition 7.1.1]. The truncation index is given by $m = (m^+, m^-)$ where $m^+ > m^-$ are two integers.

4.1. The case of G.

Definition 4.1. Let $\mathcal{M}_{0,m,G}$ denote the moduli space which associates to any scheme S over $\mathrm{Spec}(\mathbb{Z}_p)$ the set of chains $(\mathcal{W}_0 \subset \mathcal{W}_1 \subset \cdots \subset \mathcal{W}_n)$ of $\mathcal{O}_S[t]$-submodules of $\mathcal{O}_S[t, t^{-1}, (t + w_p)^{-1}]^n$ fitting into a commutative diagram with injective morphisms

$$\begin{tikzcd}
t^{m^-}\mathbb{V}_0[t]_{\mathcal{O}_S} \arrow[r] & t^{m^-}\mathbb{V}_1[t]_{\mathcal{O}_S} \arrow[r] & \cdots \arrow[r] & t^{m^-}\mathbb{V}_n[t]_{\mathcal{O}_S} \\
\mathcal{W}_0 \arrow[r] \arrow[u] & \mathcal{W}_1 \arrow[r] \arrow[u] & \cdots \arrow[r] & \mathcal{W}_n \arrow[u] \\
t^{m^+}\mathbb{V}_0[t]_{\mathcal{O}_S} \arrow[r] \arrow[u] & t^{m^+}\mathbb{V}_1[t]_{\mathcal{O}_S} \arrow[r] \arrow[u] & \cdots \arrow[r] & t^{m^+}\mathbb{V}_n[t]_{\mathcal{O}_S} \arrow[u]
\end{tikzcd}$$

where

- $\mathcal{W}_i/t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S} \subset t^{m^-}\mathbb{V}_i[t]_{\mathcal{O}_S}/t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S}$ is locally a direct factor as an $\mathcal{O}_S$-module and its $\mathcal{O}_S$-rank is independent of i, and
- $\mathcal{W}_n = (t + w_p)^{-1}\mathcal{W}_0$.

Remark 4.2. Using the natural sequence

$$\mathcal{W}_i/t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S} \longrightarrow t^{m^-}\mathbb{V}_i[t]_{\mathcal{O}_S}/t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S} \longrightarrow t^{m^-}\mathbb{V}_i[t]_{\mathcal{O}_S}/\mathcal{W}_i,$$

whose middle term is $\mathcal{O}_S$-projective, we see that

$$\mathcal{W}_i/t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S} \subset t^{m^-}\mathbb{V}_i[t]_{\mathcal{O}_S}/t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S}$$

is locally a direct factor as an $\mathcal{O}_S$-module if and only if $t^{m^-}\mathbb{V}_i[t]_{\mathcal{O}_S}/\mathcal{W}_i$ is $\mathcal{O}_S$-projective.

The moduli space $\mathcal{M}_{0,m,G}$ is shown to be representable by a projective scheme over $\mathrm{Spec}(\mathbb{Z}_p)$ in [HN, Definition 1, 2]. The scheme $\mathcal{M}_{0,m,G}$ serves as a (truncated) deformation of the affine Grassmannian $\mathrm{Gr}_{G,\mathbb{Q}_p}$ from the affine flag variety $\mathrm{Fl}_{G,\mathbb{F}_p}$, which is justified by the following lemmas. We use the equivalent definition of Gr_G (resp. Fl_G) given in Lemma 3.3 (resp. Lemma 3.4).

Over the generic point $\mathrm{Spec}(\mathbb{Q}_p)$, a chain $(\mathcal{W}_0 \subset \mathcal{W}_1 \subset \cdots \subset \mathcal{W}_n)$ as in Definition 4.1 is completely determined by $\mathcal{W}_0$, as proved in the following lemma.

Lemma 4.3. *Over the generic point $\mathbb{Q}_p$, the morphism*

$$\frac{\mathcal{W}_{i-1}}{t^{m^+}\mathbb{V}_{i-1}[t]_{\mathcal{O}_S}} \longrightarrow \frac{\mathcal{W}_i}{t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S}}$$

is an isomorphism.

Proof. It suffices to show that

$$\frac{\mathcal{W}_0}{t^{m^+}\mathbb{V}_0[t]_{\mathcal{O}_S}} \longrightarrow \frac{\mathcal{W}_n}{t^{m^+}\mathbb{V}_n[t]_{\mathcal{O}_S}}$$

is an isomorphism. Multiplication by $(t + w_p)$ induces an isomorphism on $\mathcal{W}_0/t^{m^+}\mathbb{V}_0[t]_{\mathcal{O}_S}$ since w_p is invertible in $\mathbb{Q}_p$ and

$$(t + w_p)^{-1}\mathcal{O}_S[t] \cap \mathcal{O}_S[t] = \mathcal{O}_S[t]$$

inside $\mathcal{O}_S[t, t^{-1}, (t + w_p)^{-1}]$. $\square$

Lemma 4.4. *There exists a canonical closed embedding taking the generic fiber $\mathcal{M}_{0,m,G,\eta}$ of $\mathcal{M}_{0,m,G}$ into Gr_G.*

Proof. The morphism is given by

$$(\mathcal{W}_0 \subset \mathcal{W}_1 \subset \cdots \subset \mathcal{W}_n) \longmapsto \mathcal{W}'_0,$$

where $\mathcal{W}'_0$ is the unique lattice in $\mathcal{O}_S(\!(t)\!)^n$ such that

$$t^{m^+}\mathcal{O}_S[\![t]\!]^n \subset \mathcal{W}'_0 \subset t^{m^-}\mathcal{O}_S[\![t]\!]^n \quad \text{and} \quad \mathcal{W}'_0/t^{m^+}\mathcal{O}_S[\![t]\!]^n = \mathcal{W}_0/\mathbb{V}_0[t]_{\mathcal{O}_S},$$

where, for the second equality, we used the identification

$$t^{m^-}\mathcal{O}_S[\![t]\!]^n/t^{m^+}\mathcal{O}_S[\![t]\!]^n = t^{m^-}\mathbb{V}_0[t]_{\mathcal{O}_S}/t^{m^+}\mathbb{V}_0[t]_{\mathcal{O}_S}.$$

By Lemma 4.3 and Definition 3.1(ii), this is an embedding. It identifies $\mathcal{M}_{0,m,G,\eta}$ with the image

$$\mathrm{Gr}_{G,m}(R) := \{\, R[\![t]\!]\text{-lattices } \mathfrak{L} \subset R(\!(t)\!)^n \mid t^{m^+}R[\![t]\!]^n \subset \mathfrak{L} \subset t^{m^-}R[\![t]\!]^n \,\},$$

which is a closed subscheme of Gr_G by [Zhu16, Theorem 1.1.3] (this scheme may be used to define the ind-scheme structure on Gr_G). ☐

Remark 4.5. For convenience, throughout this paper, when the embedding is canonical, we write it as an inclusion. For example, we use $\mathcal{M}_{0,m,G,\eta} \subset \mathrm{Gr}_G$ to denote the canonical embedding in Lemma 4.4.

Lemma 4.6. *There exists a canonical closed embedding taking the special fiber $\mathcal{M}_{0,m,G,s}$ of $\mathcal{M}_{0,m,G}$ into Fl_G.*

Proof. The map is

$$(\mathcal{W}_0 \subset \mathcal{W}_1 \subset \cdots \subset \mathcal{W}_n) \longmapsto (\mathcal{W}'_0 \subset \mathcal{W}'_1 \subset \cdots \subset \mathcal{W}'_n),$$

where $\mathcal{W}'_i$ is the unique lattice in $\mathcal{O}_S(\!(t)\!)^n$ such that

$$t^{m^+}\mathcal{O}_S[\![t]\!]^n \subset \mathcal{W}'_i \subset t^{m^-}\mathcal{O}_S[\![t]\!]^n \quad \text{and} \quad \mathcal{W}'_i/t^{m^+}\Lambda_{i,\mathcal{O}_S} = \mathcal{W}_i/\mathbb{V}_i[t]_{\mathcal{O}_S},$$

using the identification

$$t^{m^-}\Lambda_{i,\mathcal{O}_S}/t^{m^+}\Lambda_{i,\mathcal{O}_S} = t^{m^-}\mathbb{V}_i[t]_{\mathcal{O}_S}/t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S}.$$

By Definition 3.1(ii), this is an embedding. It is closed since $\mathcal{M}_{0,m,G,s}$ is proper ([HN]) and Fl_G is ind-proper ([Ri16, Theorem A]); we use that any proper monomorphism is a closed embedding [StaPro, Tag 04XV]. ☐

Remark 4.7. Moreover, one can show that there exists a closed embedding integrally $\mathcal{M}_{0,m,G} \subset \mathrm{Gr}_{\mathcal{I},w_p}$, where the target is the specialization at w_p of the Pappas–Zhu Grassmannian [PZ] attached to a smooth affine group scheme $\mathcal{I}$ over the affine line $\mathrm{Spec}(\mathbb{Z}_p[u])$. This fact will not be used in this paper; see [HLS, Sec. 9.5] for details.

4.2. **The case of T.** Using the definition of $\mathcal{M}_{0,m,\mathrm{GL}_1}$, we define the truncated deformations in the case of T as a direct product.

Definition 4.8. Let $\mathcal{M}_{0,m,T}$ denote the direct product of n copies of $\mathcal{M}_{0,m,\mathrm{GL}_1}$. For any scheme S over $\mathrm{Spec}(\mathbb{Z}_p)$, the S-valued points of $\mathcal{M}_{0,m,T}$ are given by n-tuples $(\mathcal{U}_1, \mathcal{U}_2, \cdots, \mathcal{U}_n)$ such that

- $t^{m^+}\mathcal{O}_S[t] \subset \mathcal{U}_i \subset t^{m^-}\mathcal{O}_S[t]$, and
- $\mathcal{U}_i/t^{m^+}\mathcal{O}_S[t] \subset t^{m^-}\mathcal{O}_S[t]/t^{m^+}\mathcal{O}_S[t]$ is locally a direct factor as an $\mathcal{O}_S$-module.

By taking $\mathcal{W}_i := (t+w_p)^{-1}\mathcal{U}_1 \oplus \cdots \oplus (t+w_p)^{-1}\mathcal{U}_i \oplus \mathcal{U}_{i+1} \oplus \cdots \oplus \mathcal{U}_n$, we can embed $\mathcal{M}_{0,m,T}$ into $\mathcal{M}_{0,m,G}$. As both sides are proper, this is a closed embedding by [StaPro, Tag 04XV].

The scheme $\mathcal{M}_{0,m,T}$ also serves as a truncated deformation.

Lemma 4.9. *There exists a canonical closed embedding taking the generic fiber $\mathcal{M}_{0,m,T,\eta}$ of $\mathcal{M}_{0,m,T}$ into Gr_T. The embeddings $\mathcal{M}_{0,m,T,\eta} \subset \mathrm{Gr}_T$, $\mathcal{M}_{0,m,T} \subset \mathcal{M}_{0,m,G}$, $\mathcal{M}_{0,m,G,\eta} \subset \mathrm{Gr}_G$ and $\mathrm{Gr}_T \subset \mathrm{Gr}_G$ (induced from $T \subset G$) form a Cartesian diagram*

$$\begin{array}{ccc} \mathcal{M}_{0,m,T,\eta} & \longrightarrow & \mathrm{Gr}_T \\ \big\downarrow & & \big\downarrow \\ \mathcal{M}_{0,m,G,\eta} & \longrightarrow & \mathrm{Gr}_G. \end{array}$$

Proof. The affine Grassmannian Gr_T is, by definition, also a direct product of n copies of $\mathrm{Gr}_{\mathrm{GL}_1}$. The existence of the embedding is a direct consequence of Lemma 4.4 applied to GL_1. For $a = (a_i)_i \in T(R(\!(t)\!))$, we have

$$a\Lambda_{0,R} = a_1 R[\![t]\!] \oplus \cdots \oplus a_n R[\![t]\!],$$

so the Cartesian property follows. □

Lemma 4.10. *There exists a canonical closed embedding taking the special fiber $\mathcal{M}_{0,m,T,s}$ of $\mathcal{M}_{0,m,T}$ into $\mathrm{Fl}_T = \mathrm{Gr}_T$. The embeddings $\mathcal{M}_{0,m,T,s} \subset \mathrm{Fl}_T$, $\mathcal{M}_{0,m,T} \subset \mathcal{M}_{0,m,G}$, $\mathcal{M}_{0,m,G,s} \subset \mathrm{Fl}_G$ and $\mathrm{Fl}_T \subset \mathrm{Fl}_G$ form a Cartesian diagram*

$$\begin{tikzcd} \mathcal{M}_{0,m,T,s} \arrow[r] \arrow[d] & \mathrm{Fl}_T \arrow[d] \\ \mathcal{M}_{0,m,G,s} \arrow[r] & \mathrm{Fl}_G. \end{tikzcd}$$

Proof. Since $\mathrm{Fl}_T = \mathrm{Gr}_T$, the definition of this closed embedding is the same as in Lemma 4.9. For $a = (a_i)_i \in T(R(\!(t)\!))$, we have

$$a\Lambda_{i,R} = t^{-1}a_1 R[\![t]\!] \oplus \cdots \oplus t^{-1}a_i R[\![t]\!] \oplus a_{i+1} R[\![t]\!] \oplus \cdots \oplus a_n R[\![t]\!],$$

so the Cartesian property follows. □

4.3. **The case of B.** Now we give the analogue of $\mathcal{M}_{0,m,G}$ for B, whose definition is more subtle.

Remark 4.11. Throughout this paper, when it is clear from the context, for any ring R, we identify R^{b_1} with the subset of R^{b_2} ($b_1 \le b_2$) whose last $b_2 - b_1$ coordinates are zero, and in this case, we use the notation $R^{b_1} \subset R^{b_2}$.

Definition 4.12. Let $\mathcal{M}_{0,m,B}$ denote the moduli space which associates to any scheme S over $\mathrm{Spec}(\mathbb{Z}_p)$ the set of $\mathcal{O}_S[t]$-modules $t^{m^+}\mathbb{V}_0[t]_{\mathcal{O}_S} \subset \mathcal{W}_0 \subset t^{m^-}\mathbb{V}_0[t]_{\mathcal{O}_S}$ such that

- $\mathcal{W}_0/t^{m^+}\mathbb{V}_0[t]_{\mathcal{O}_S} \subset t^{m^-}\mathbb{V}_0[t]_{\mathcal{O}_S}/t^{m^+}\mathbb{V}_0[t]_{\mathcal{O}_S}$ is locally a direct factor as an $\mathcal{O}_S$-module, and
- for $\mathcal{W}_{0,i} := \mathcal{W}_0 \cap \mathcal{O}_S(t)^i$ (see Remark 4.11; in particular, $\mathcal{W}_{0,0} = 0$ and $\mathcal{W}_{0,n} = \mathcal{W}_0$), there is an extension $\mathcal{W}_{0,i-1} \to \mathcal{W}_{0,i} \to \mathcal{Q}_i$ where the first arrow is the natural inclusion and the second arrow is the projection to the i-th component such that, as the image of the projection, the $\mathcal{O}_S[t]$-module $t^{m^+}\mathcal{O}_S[t] \subset \mathcal{Q}_i \subset t^{m^-}\mathcal{O}_S[t]$ is a well-defined point in $\mathcal{M}_{0,m,\mathrm{GL}_1}(S)$ (in other words, $\mathcal{Q}_i/t^{m^+}\mathcal{O}_S[t] \subset t^{m^-}\mathcal{O}_S[t]/t^{m^+}\mathcal{O}_S[t]$ is locally a direct factor as an $\mathcal{O}_S$-module).

The moduli space $\mathcal{M}_{0,m,B}$ is not defined in [HN] or [HLS]. To show the representability of $\mathcal{M}_{0,m,B}$, we need the following lemmas.

Lemma 4.13. *In Definition 4.12, for each i, the quotient*

$$\mathcal{W}_{0,i}/t^{m^+}\mathcal{O}_S[t]^i \subset t^{m^-}\mathcal{O}_S[t]^i/t^{m^+}\mathcal{O}_S[t]^i$$

is locally a direct factor as an $\mathcal{O}_S$-module.

Proof. From the definition of $\mathcal{W}_{0,i}$, it is clear that $t^{m^+}\mathcal{O}_S[t]^i \subset \mathcal{W}_{0,i} \subset t^{m^-}\mathcal{O}_S[t]^i$, so the inclusion above makes sense. To show that it is locally a direct factor, it suffices to prove that $t^{m^-}\mathcal{O}_S[t]^i/\mathcal{W}_{0,i}$ is a projective $\mathcal{O}_S$-module (see Remark 4.2). For $i = 1$, this is given in the definition, since $\mathcal{W}_{0,1} = \mathcal{Q}_1$ is required to be a well-defined point in $\mathcal{M}_{0,m,\mathrm{GL}_1}$. For larger i, we have extensions

$$t^{m^-}\mathcal{O}_S[t]^{i-1}/\mathcal{W}_{0,i-1} \longrightarrow t^{m^-}\mathcal{O}_S[t]^i/\mathcal{W}_{0,i} \longrightarrow t^{m^-}\mathcal{O}_S[t]/\mathcal{Q}_i,$$

and the result follows by induction on i. □

Lemma 4.14. *In Definition 4.12, for each i, the natural $\mathcal{O}_S$-map*

$$t^{m^-}\mathcal{O}_S[t]^{i-1}/\mathcal{W}_{0,i-1} \longrightarrow t^{m^-}\mathcal{O}_S[t]^i/\mathcal{W}_{0,i}$$

is universally injective, in the sense that after any base change $S' \to S$, the induced $\mathcal{O}_{S'}$-map is injective.

Proof. Taking quotients, the extension of $\mathcal{O}_S[t]$-modules $\mathcal{W}_{0,i-1} \to \mathcal{W}_{0,i} \to \mathcal{Q}_i$ induces an extension of $\mathcal{O}_S$-modules

$$\mathcal{W}_{0,i-1}/t^{m^+}\mathcal{O}_S[t]^{i-1} \longrightarrow \mathcal{W}_{0,i}/t^{m^+}\mathcal{O}_S[t]^i \longrightarrow \mathcal{Q}_i/t^{m^+}\mathcal{O}_S[t],$$

where the second arrow is still the projection to the i-th component of $t^{m^-}\mathcal{O}_S[t]^i/t^{m^+}\mathcal{O}_S[t]^i$. This extension splits since $\mathcal{Q}_i/t^{m^+}\mathcal{O}_S[t]$ is $\mathcal{O}_S$-projective.

After base change along $S' \to S$, we obtain an extension

$$\mathcal{W}'_{0,i-1}/t^{m^+}\mathcal{O}_{S'}[t]^{i-1} \longrightarrow \mathcal{W}'_{0,i}/t^{m^+}\mathcal{O}_{S'}[t]^i \longrightarrow \mathcal{Q}'_i/t^{m^+}\mathcal{O}_{S'}[t],$$

where the second arrow is again the projection to the i-th component of $t^{m^-}\mathcal{O}_{S'}[t]^i/t^{m^+}\mathcal{O}_{S'}[t]^i$, and $\mathcal{W}'_{0,i}$ (resp. $\mathcal{Q}'_i$) is the image of $\mathcal{W}_{0,i} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'} \to t^{m^-}\mathcal{O}_{S'}[t]^i$ (resp. $\mathcal{Q}_i \otimes_{\mathcal{O}_S} \mathcal{O}_{S'} \to t^{m^-}\mathcal{O}_{S'}[t]$). This implies

$$\mathcal{W}'_{0,i} \cap t^{m^-}\mathcal{O}_{S'}[t]^{i-1} = \mathcal{W}'_{0,i-1},$$

which is exactly the desired universal injectivity. $\square$

Remark 4.15. As a universally injective $\mathcal{O}_S$-map,

$$t^{m^-}\mathcal{O}_S[t]^{i-1}/\mathcal{W}_{0,i-1} \longrightarrow t^{m^-}\mathcal{O}_S[t]^i/\mathcal{W}_{0,i}$$

is, in particular, injective, which implies $\mathcal{W}_{0,i} \cap t^{m^-}\mathcal{O}_S[t]^{i-1} = \mathcal{W}_{0,i-1}$; this recovers the equality $\mathcal{W}_{0,i} \cap \mathcal{O}_S(t)^{i-1} = \mathcal{W}_{0,i-1}$. The universal injectivity is useful in proving the representability of $\mathcal{M}_{0,m,B}$, since it defines an open condition.

Proposition 4.16. *The moduli space $\mathcal{M}_{0,m,B}$ is representable by a scheme.*

Proof. By Lemma 4.13, there is a natural embedding

$$\begin{aligned}\mathcal{M}_{0,m,B} \hookrightarrow &\prod_{1\le i\le n}\Big(\mathrm{gr}\big(t^{m^-}\mathbb{Z}_p[t]^i/t^{m^+}\mathbb{Z}_p[t]^i\big) \times \mathrm{gr}\big((t^{m^-}\mathbb{Z}_p[t]/t^{m^+}\mathbb{Z}_p[t])_i\big)\Big)\\ &= \prod_{1\le i\le n}\Big(\mathrm{gr}\big(\mathbb{Z}_p^{i(m^+-m^-)}\big) \times \mathrm{gr}\big((\mathbb{Z}_p^{m^+-m^-})_i\big)\Big).\end{aligned}$$

where $\mathrm{gr}(\mathbb{Z}_p^{i(m^+-m^-)})$ is the classical Grassmannian scheme (strictly speaking, a disjoint union of classcial Grassmannian schemes indexed by ranks) whose $S/\mathrm{Spec}(\mathbb{Z}_p)$-valued points are locally $\mathcal{O}_S$-direct factors $L \subset \mathcal{O}_S^{i(m^+-m^-)}$, and $(t^{m^-}\mathbb{Z}_p[t]/t^{m^+}\mathbb{Z}_p[t])_i$ (resp. $(\mathbb{Z}_p^{m^+-m^-})_i$) is a copy of $t^{m^-}\mathbb{Z}_p[t]/t^{m^+}\mathbb{Z}_p[t]$ (resp. $\mathbb{Z}_p^{m^+-m^-}$). Choose isomorphisms

$$t^{m^-}\mathbb{Z}_p[t]^i/t^{m^+}\mathbb{Z}_p[t]^i \cong \mathbb{Z}_p^{i(m^+-m^-)}$$

such that

$$t^{m^-}\mathbb{Z}_p[t]^{i-1}/t^{m^+}\mathbb{Z}_p[t]^{i-1} \subset t^{m^-}\mathbb{Z}_p[t]^i/t^{m^+}\mathbb{Z}_p[t]^i$$

corresponds to

$$\mathbb{Z}_p^{(i-1)(m^+-m^-)} \subset \mathbb{Z}_p^{i(m^+-m^-)},$$

and, viewing $(t^{m^-}\mathbb{Z}_p[t]/t^{m^+}\mathbb{Z}_p[t])_i \subset t^{m^-}\mathbb{Z}_p[t]^i/t^{m^+}\mathbb{Z}_p[t]^i$ as the subset where the first $i-1$ coordinates vanish, identify it with $(\mathbb{Z}_p^{m^+-m^-})_i \subset \mathbb{Z}_p^{i(m^+-m^-)}$ where the first $(i-1)(m^+-m^-)$ coordinates vanish.

Under this embedding $\mathcal{W}_0 \mapsto (W_i, Q_i)_i$, the submodule W_i is the image of $\mathcal{W}_{0,i}/t^{m^+}\mathcal{O}_S[t]^i$ in $t^{m^-}\mathcal{O}_S[t]^i/t^{m^+}\mathcal{O}_S[t]^i = \mathcal{O}_S^{i(m^+-m^-)}$, and Q_i is the image of $\mathcal{Q}_i/t^{m^+}\mathcal{O}_S[t]$ in $(t^{m^-}\mathbb{Z}_p[t]/t^{m^+}\mathbb{Z}_p[t])_i = (\mathbb{Z}_p^{m^+-m^-})_i$.

The image is the locally closed subscheme of

$$\prod_{1\le i\le n}\big(\mathrm{gr}(\mathbb{Z}_p^{i(m^+-m^-)}) \times \mathrm{gr}((\mathbb{Z}_p^{m^+-m^-})_i)\big)$$

cut out by:

- the closed condition $W_i \subset \mathcal{O}_S^{i(m^+-m^-)}$;
- the closed condition $W_i \subset W_n$;

- the open condition that $\mathcal{O}_S^{i(m^+-m^-)}/W_i \to \mathcal{O}_S^{n(m^+-m^-)}/W_n$ is universally injective (see Lemma 4.14 and Remark 4.15);
- the closed condition $W_n/W_i \subset Q_i$ in $(\mathcal{O}_S^{m^+-m^-})_i$;
- the open condition that the inclusion $W_n/W_i \subset Q_i$ is surjective; and
- the closed condition that W_n is stable under the nilpotent t-action on

$$t^{m^-}\mathcal{O}_S[t]^n/t^{m^+}\mathcal{O}_S[t]^n \cong \mathcal{O}_S^{n(m^+-m^-)}.$$

Hence $\mathcal{M}_{0,m,B}$ is representable by a locally closed subscheme of a projective scheme. □

Lemma 4.17. *There exist natural embeddings $\mathcal{M}_{0,m,T} \subset \mathcal{M}_{0,m,B} \subset \mathcal{M}_{0,m,G}$ such that the composite $\mathcal{M}_{0,m,T} \subset \mathcal{M}_{0,m,G}$ is the same as the embedding given in Definition 4.8.*

Proof. The embedding $\mathcal{M}_{0,m,T} \subset \mathcal{M}_{0,m,B}$ is given by

$$(\mathcal{U}_\bullet) \longmapsto \mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_n =: \mathcal{W}_0,$$

which clearly defines a point of $\mathcal{M}_{0,m,B}$.

For the embedding $\mathcal{M}_{0,m,B} \subset \mathcal{M}_{0,m,G}$, define

$$\mathcal{W}_i := \mathcal{W}_0 + (t+w_p)^{-1}\mathcal{W}_{0,i}.$$

Then $t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S} \subset \mathcal{W}_i \subset t^{m^-}\mathbb{V}_i[t]_{\mathcal{O}_S}$, and it remains to show that $t^{m^-}\mathbb{V}_i[t]_{\mathcal{O}_S}/\mathcal{W}_i$ is a projective $\mathcal{O}_S$-module whose rank is independent of i. For $i = 0, n$, projectivity follows from the definition. For other i, first observe a natural isomorphism

$$\mathcal{W}_i/\mathcal{W}_{i-1} \cong (t+w_p)^{-1}\mathcal{Q}_i/\mathcal{Q}_i.$$

Indeed,

$$\frac{\mathcal{W}_i}{\mathcal{W}_{i-1}} = \frac{\mathcal{W}_0 + (t+w_p)^{-1}\mathcal{W}_{0,i}}{\mathcal{W}_0 + (t+w_p)^{-1}\mathcal{W}_{0,i-1}} = \frac{(t+w_p)^{-1}\mathcal{W}_{0,i}}{(t+w_p)^{-1}\mathcal{W}_{0,i-1} + \mathcal{W}_{0,i}} = \frac{(t+w_p)^{-1}\mathcal{Q}_i}{\mathcal{Q}_i},$$

where we used $(t+w_p)^{-1}\mathcal{O}_S[t] \cap \mathcal{O}_S[t] = \mathcal{O}_S[t]$. In particular, $\mathcal{W}_i/\mathcal{W}_{i-1}$ is a rank-1 projective $\mathcal{O}_S$-module.

Assume $t^{m^-}\mathbb{V}_i[t]_{\mathcal{O}_S}/\mathcal{W}_i$ is $\mathcal{O}_S$-projective. From the extension

$$\mathcal{W}_i/\mathcal{W}_{i-1} \longrightarrow t^{m^-}\mathbb{V}_i[t]_{\mathcal{O}_S}/\mathcal{W}_{i-1} \longrightarrow t^{m^-}\mathbb{V}_i[t]_{\mathcal{O}_S}/\mathcal{W}_i,$$

we deduce that $t^{m^-}\mathbb{V}_i[t]_{\mathcal{O}_S}/\mathcal{W}_{i-1}$ is $\mathcal{O}_S$-projective. Now, from

$$t^{m^-}\mathbb{V}_{i-1}[t]_{\mathcal{O}_S}/\mathcal{W}_{i-1} \longrightarrow t^{m^-}\mathbb{V}_i[t]_{\mathcal{O}_S}/\mathcal{W}_{i-1} \longrightarrow t^{m^-}\mathbb{V}_i[t]_{\mathcal{O}_S}/t^{m^-}\mathbb{V}_{i-1}[t]_{\mathcal{O}_S},$$

we see that $t^{m^-}\mathbb{V}_{i-1}[t]_{\mathcal{O}_S}/\mathcal{W}_{i-1}$ is $\mathcal{O}_S$-projective; thus, by downward induction on i, projectivity holds for all i. A rank count using the two extensions above shows that the $\mathcal{O}_S$-rank of $t^{m^-}\mathbb{V}_i[t]_{\mathcal{O}_S}/\mathcal{W}_i$ is independent of i.

Finally, by construction, the composite $\mathcal{M}_{0,m,T} \subset \mathcal{M}_{0,m,G}$ is the same as in Definition 4.8. □

Remark 4.18. Using the proof of Proposition 4.16 and the construction of the embedding $\mathcal{M}_{0,m,B} \subset \mathcal{M}_{0,m,G}$ in Lemma 4.17, one can show that $\mathcal{M}_{0,m,B}$ is a locally closed subscheme of $\mathcal{M}_{0,m,G}$ under that embedding. Note that the additional requirement $\mathcal{W}_i = \mathcal{W}_0 + (t+w_p)^{-1}\mathcal{W}_{0,i}$ defines a locally closed condition. Since we do not need this result in this paper, the proof is omitted.

We now justify the definition of $\mathcal{M}_{0,m,B}$ by showing that it is a truncated deformation of $\mathrm{Gr}_{B,\mathbb{Q}_p}$ from $\mathrm{Fl}_{B,\mathbb{F}_p}$. Note that in our setting, $\mathrm{Gr}_{B,k} = \mathrm{Fl}_{B,k}$. First, we give equivalent characterizations of $\mathrm{Gr}_{B,k}$.

Lemma 4.19. *Let R be any k-algebra. The set $\mathrm{Gr}_{B,k}(R)$ identifies functorially with the set of isomorphism classes of pairs $(\mathcal{F}, \alpha)$ where $\mathcal{F}$ is a fpqc B-torsor over the formal disc $\mathbb{D}_R := \mathrm{Spec}(R[\![t]\!])$ and α is a trivialization of $\mathcal{F}$ over the punctured formal disc $\mathbb{D}_R^* := \mathrm{Spec}(R(\!(t)\!))$.*

Proof. This is a special case of [Zhu16, Proposition 1.3.6]. The map is induced by

$$g \in B(R(\!(t)\!)) \longmapsto g(\mathcal{F}_0, \alpha_0),$$

where $\mathcal{F}_0 = B \times \mathbb{D}_R$ and α_0 is the identity morphism. The main point is that any such $\mathcal{F}$ is étale locally on $\mathrm{Spec}(R)$ trivial [Zhu16, Lemma 1.3.7]. □

We also give an equivalent characterization of $\mathrm{Gr}_{B,k}$ using chains of lattices with additional structure.

Lemma 4.20. *Let R be any k-algebra. The set $\mathrm{Gr}_{B,k}(R)$ identifies functorially with the set of chains of projective $R[\![t]\!]$-modules $\mathcal{C}_0 \subset \mathcal{C}_1 \subset \cdots \subset \mathcal{C}_n$ such that $\mathcal{C}_i$ is an $R[\![t]\!]$-lattice in $R(\!(t)\!)^i \subset R(\!(t)\!)^n$ and $\mathcal{C}_i = \mathcal{C}_n \cap R(\!(t)\!)^i$, with the consecutive quotient $\mathcal{C}_i/\mathcal{C}_{i-1}$ being an $R[\![t]\!]$-lattice in $R(\!(t)\!) = R(\!(t)\!)^i/R(\!(t)\!)^{i-1}$ via the projection to the i-th component.*

Proof. By Lemma 4.3, $\mathrm{Gr}_B(R)$ consists of isomorphism classes of pairs $(\mathcal{F}, \alpha)$. Starting from $(\mathcal{F}, \alpha)$, note that B stabilizes the standard flag $0 \subset R[\![t]\!] \subset \cdots \subset R[\![t]\!]^n$. Using the contracted products $\mathcal{V}_i := \mathcal{F} \times^B \mathcal{O}^i_{R[\![t]\!]}$, which are rank-$i$ vector bundles on $\mathbb{D}_R$ [Zhu16, Sec. 0.3.3], taking global sections yields a chain $\mathcal{C}_\bullet$ of projective $R[\![t]\!]$-modules with $\mathcal{C}_i := \Gamma(\mathcal{V}_i)$ of rank i. Via α we have

$$\Gamma(\mathcal{F} \times^B \mathcal{O}^i_{R[\![t]\!]}) \subset \Gamma(\mathcal{F} \times^B \mathcal{O}^i_{R(\!(t)\!)}) \xrightarrow{\alpha} \Gamma(\mathcal{O}^i_{R(\!(t)\!)}) = R(\!(t)\!)^i \subset R(\!(t)\!)^n,$$

so each $\mathcal{C}_i$ is an $R[\![t]\!]$-lattice in $R(\!(t)\!)^i$ and $\mathcal{C}_i = \mathcal{C}_n \cap R(\!(t)\!)^i$. Moreover, $\mathcal{C}_i/\mathcal{C}_{i-1}$ is the $R[\![t]\!]$-lattice in $R(\!(t)\!)$ corresponding to $\mathcal{F} \times^B (\mathcal{O}^i_{R[\![t]\!]}/\mathcal{O}^{i-1}_{R[\![t]\!]})$.

Conversely, from such a chain $\mathcal{C}_\bullet$ one obtains a B-torsor as the isomorphism scheme between $\mathcal{C}_\bullet$ and the standard chain $0 \subset R[\![t]\!] \subset \cdots \subset R[\![t]\!]^n$. This is a B-torsor since the automorphism group scheme of the standard chain is B and, as $\mathcal{C}_i$ and $\mathcal{C}_i/\mathcal{C}_{i-1}$ are étale locally free on $\mathrm{Spec}(R)$ [Zhu16, Lemma 1.3.7], the chain is étale locally isomorphic to the standard one. The trivialization α is induced from $\mathcal{C}_i[\frac{1}{t}] = R(\!(t)\!)^i$.

It is standard to check that these constructions are inverse to each other. □

Lemma 4.20 will be used as the definition of $\mathrm{Gr}_{B,k}$ in the next two lemmas.

Lemma 4.21. *There exists a natural closed embedding taking the generic fiber $\mathcal{M}_{0,m,B,\eta}$ of $\mathcal{M}_{0,m,B}$ into Gr_B. The embeddings $\mathcal{M}_{0,m,B,\eta} \subset \mathrm{Gr}_B$, $\mathcal{M}_{0,m,B} \subset \mathcal{M}_{0,m,G}$, $\mathcal{M}_{0,m,G,\eta} \subset \mathrm{Gr}_G$ and $\mathrm{Gr}_B \subset \mathrm{Gr}_G$ (induced from $B \subset G$) form a Cartesian diagram*

$$\begin{tikzcd} \mathcal{M}_{0,m,B,\eta} \arrow[r] \arrow[d] & \mathrm{Gr}_B \arrow[d] \\ \mathcal{M}_{0,m,G,\eta} \arrow[r] & \mathrm{Gr}_G. \end{tikzcd}$$

Proof. Let R be a $\mathbb{Q}_p$-algebra and $\mathcal{W}_0 \in \mathcal{M}_{0,m,B,\eta}(R)$. To define the embedding $\mathcal{M}_{0,m,B,\eta} \subset \mathrm{Gr}_B$, specify the image of $\mathcal{W}_0$ as follows. Define

$$t^{m^+} R[\![t]\!]^i \subset \mathcal{C}_i \subset t^{m^-} R[\![t]\!]^i$$

to be the $R[\![t]\!]$-lattice given by $\mathcal{W}_{0,i}/t^{m^+}R[t]^i$ using Definition 3.1(i). This yields a point in $\mathrm{Gr}_B(R)$ since

$$\begin{aligned} \mathcal{W}_{0,n} \cap R(t)^i = \mathcal{W}_{0,i} &\Longleftrightarrow \frac{\mathcal{W}_{0,n}}{t^{m^+}R[t]^n} \cap \frac{t^{m^-}R[t]^i}{t^{m^+}R[t]^i} = \frac{\mathcal{W}_{0,i}}{t^{m^+}R[t]^i} \\ &\Longleftrightarrow \frac{\mathcal{C}_n}{t^{m^+}R[\![t]\!]^n} \cap \frac{t^{m^-}R[\![t]\!]^i}{t^{m^+}R[\![t]\!]^i} = \frac{\mathcal{C}_i}{t^{m^+}R[\![t]\!]^i} \\ &\Longleftrightarrow \mathcal{C}_n \cap R(\!(t)\!)^i = \mathcal{C}_i, \end{aligned}$$

and the fact that $\mathcal{C}_i/\mathcal{C}_{i-1}$ is an $R[\![t]\!]$-lattice follows from

$$\frac{t^{m^-}R(\!(t)\!)}{\mathcal{C}_i/\mathcal{C}_{i-1}} = \frac{\frac{t^{m^-}R[\![t]\!]}{t^{m^+}R[\![t]\!]}}{\frac{\mathcal{C}_i}{t^{m^+}R[\![t]\!]^i} / \frac{\mathcal{C}_{i-1}}{t^{m^+}R[\![t]\!]^{i-1}}} = \frac{\frac{t^{m^-}R[t]}{t^{m^+}R[t]}}{\frac{\mathcal{W}_{0,i}}{t^{m^+}R[t]^i} / \frac{\mathcal{W}_{0,i-1}}{t^{m^+}R[t]^{i-1}}} = \frac{t^{m^-}R[t]}{\mathcal{Q}_i}.$$

Since $\mathrm{Gr}_B \subset \mathrm{Gr}_G$ is given by

$$\mathcal{C}_0 \subset \mathcal{C}_1 \subset \cdots \subset \mathcal{C}_n \longmapsto \mathcal{C}_n,$$

the diagram is Cartesian, and thus the map $\mathcal{M}_{0,m,B,\eta} \to \mathrm{Gr}_B$ is a closed embedding by Lemma 4.4. □

Lemma 4.22. *There exists a canonical closed embedding taking the special fiber $\mathcal{M}_{0,m,B,s}$ of $\mathcal{M}_{0,m,B}$ into $\mathrm{Fl}_B = \mathrm{Gr}_B$. The embeddings $\mathcal{M}_{0,m,B,s} \subset \mathrm{Fl}_B$, $\mathcal{M}_{0,m,B} \subset \mathcal{M}_{0,m,G}$, $\mathcal{M}_{0,m,G,s} \subset \mathrm{Fl}_G$ and $\mathrm{Fl}_B \subset \mathrm{Fl}_G$ form a Cartesian diagram*

$$\begin{tikzcd} \mathcal{M}_{0,m,B,s} \arrow[r] \arrow[d] & \mathrm{Fl}_B \arrow[d] \\ \mathcal{M}_{0,m,G,s} \arrow[r] & \mathrm{Fl}_G. \end{tikzcd}$$

Proof. The proof is the same as that of Lemma 4.21 (the base field is irrelevant). Note that $\mathrm{Fl}_B \subset \mathrm{Fl}_G$ is given by

$$\mathcal{C}_0 \subset \mathcal{C}_1 \subset \cdots \subset \mathcal{C}_n \longmapsto \mathfrak{L}_0 \subset \mathfrak{L}_1 \subset \cdots \subset \mathfrak{L}_n = t^{-1}\mathfrak{L}_0,$$

where $\mathfrak{L}_i := \mathcal{C}_n + t^{-1}\mathcal{C}_i$. Replacing $\mathcal{W}_{0,i}$ with $\mathcal{C}_i$ and working with the power series ring instead, the proof of Lemma 4.17 shows that

$$\mathfrak{L}_0 \subset \mathfrak{L}_1 \subset \cdots \subset \mathfrak{L}_n = t^{-1}\mathfrak{L}_0$$

is a well-defined t-periodic lattice chain. □

By Lemma 4.21 and Lemma 4.22, we conclude that $\mathcal{M}_{0,m,B}$ serves as a truncated deformation of $\mathrm{Gr}_{B,\mathbb{Q}_p}$ from $\mathrm{Fl}_{B,\mathbb{F}_p}$.

5. The $\mathbb{G}_m$-action and hyperbolic localization for the Iwahori level

We follow the convention that $\mathbb{G}_m$ denotes the multiplicative group scheme, which is the same as GL_1.

The consecutive quotients $\mathcal{Q}_i$ in Definition 4.12 define a morphism $\mathcal{M}_{0,m,B} \to \mathcal{M}_{0,m,T}$. Combining this morphism with the inclusion $\mathcal{M}_{0,m,B} \subset \mathcal{M}_{0,m,G}$ in Lemma 4.17, we obtain the diagram

(5.1)
$$\begin{tikzcd} & \mathcal{M}_{0,m,B} \arrow[dl] \arrow[dr] & \\ \mathcal{M}_{0,m,T} & & \mathcal{M}_{0,m,G}. \end{tikzcd}$$

In this section, we describe Diagram 5.1 via hyperbolic localization on $\mathcal{M}_{0,m,G}$. It turns out to be an analogue of [HR21, (1.1)].

Remark 5.1. We can define a morphism $\mathrm{Gr}_B \to \mathrm{Gr}_T$ by

$$\mathcal{C}_0 \subset \mathcal{C}_1 \subset \cdots \subset \mathcal{C}_n \longmapsto (\mathcal{C}_1/\mathcal{C}_0,\, \mathcal{C}_2/\mathcal{C}_1,\, \ldots,\, \mathcal{C}_n/\mathcal{C}_{n-1}).$$

Unlike Lemma 4.9 and Lemma 4.21, the commutative diagram formed by $\mathcal{M}_{0,m,B} \to \mathcal{M}_{0,m,T}$, their embeddings into the affine Grassmannians, and the map $\mathrm{Gr}_B \to \mathrm{Gr}_T$ (induced from the quotient map $B \to T$),

$$\begin{tikzcd} \mathcal{M}_{0,m,B,\eta} \arrow[r] \arrow[d] & \mathrm{Gr}_B \arrow[d] \\ \mathcal{M}_{0,m,T,\eta} \arrow[r] & \mathrm{Gr}_T, \end{tikzcd}$$

is **not** Cartesian. The reason is that $\mathcal{M}_{0,m,T}$ only requires the consecutive quotients $\mathcal{Q}_i$ to be bounded by m, while the lattices $\mathcal{W}_{0,i}$ themselves need not be bounded by m.

5.1. **The $\mathbb{G}_m$-action on truncated deformations.** To apply the techniques of [Ri19] to $\mathcal{M}_{0,m,G}$, we introduce a $\mathbb{G}_m$-action on $\mathcal{M}_{0,m,G}$, inherited from an action of a larger group scheme I_m that can be viewed as a truncated (in the sense of quotients) deformation of positive loop group schemes.

Recall the definition of I_m from [HLS, Definition 6.4.1]. Although [HLS, Definition 6.4.1] is stated for GSp_{2g}, all results in [HLS, Sec. 6.4] apply to GL_n after ignoring the symplectic form.

Definition 5.2. Let I_m be the group-valued functor assigning to a $\mathbb{Z}_p$-algebra R the group of automorphisms

$$g \in \mathrm{Aut}_{R[t]}\Big(\frac{(t+w_p)^{-1}t^{m^-}\mathbb{V}_0[t]_R}{t^{m^+}\mathbb{V}_0[t]_R}\Big)$$

with the following property:

- g preserves the images $t^{m^-}\overline{\mathbb{V}}_i[t]_R$ of $t^{m^-}\mathbb{V}_i[t]_R$ in $\frac{(t+w_p)^{-1}t^{m^-}\mathbb{V}_0[t]_R}{t^{m^+}\mathbb{V}_0[t]_R}$, for $0 \le i \le n$.

The group scheme I_m is smooth over $\mathrm{Spec}(\mathbb{Z}_p)$ with connected geometric fibers [HLS, Proposition 6.4.6]; it serves as a truncated deformation from the Iwahori subgroup to the hyperspecial subgroup [HLS, Lemma 6.2.3, Lemma 6.4.3, Remark 6.4.5].

For any $\mathbb{Z}_p$-algebra R, the group $T(R)$ acts naturally on $(t+w_p)^{-1}t^{m^-}\mathbb{V}_0[t]_R$ by

$$(t_1,\dots,t_n)\cdot(v_1,\dots,v_n) = (t_1v_1,\dots,t_nv_n),$$

and this action preserves $t^{m^-}\mathbb{V}_i[t]_R$. Hence we obtain a natural embedding $T \subset I_m$ and thus a T-action on $\mathcal{M}_{0,m,G}$. Throughout the paper, fix a cocharacter $\lambda \in X_*(T)$ such that, via the conjugation action (using $T \subset G$), the corresponding cocharacter of G yields the attractor $G^+ = B$ as in [HR21, Sec. 3.3.1]. (We recall the definitions of attractors and other objects from hyperbolic localization below and equivalently, we require that the natural pairing between λ and any positive root attached to B give a positive number.) The existence of such a cocharacter is well known for reductive groups over a field (e.g. [Mil, Theorem 25.1]); for split reductive groups (such as GL_n and GSp_{2g}), the same cocharacter defined over $\mathrm{Spec}(\mathbb{Z})$ works over any base field. Over $\mathrm{Spec}(\mathbb{Z}_p)$, the group schemes B and G^+ coincide, since both are smooth (hence reduced [StaPro, Tag 034E]) closed subgroup schemes of G and agree on every geometric fiber [HR21, Lemma 4.5].

Combining the T-action on $\mathcal{M}_{0,m,G}$ with the cocharacter λ yields a $\mathbb{G}_m$-action on $\mathcal{M}_{0,m,G}$. Considering the embeddings $\mathcal{M}_{0,m,T} \subset \mathcal{M}_{0,m,B} \subset \mathcal{M}_{0,m,G}$ from Lemma 4.17, and since the T-action is given by scalar multiplications on the coordinates of $\mathbb{V}$, we see that $\mathcal{M}_{0,m,T}$ and $\mathcal{M}_{0,m,B}$ are T-stable (though not I_m-stable). Hence they inherit $\mathbb{G}_m$-actions. The $\mathbb{G}_m$-action on $\mathcal{M}_{0,m,T}$ is easily seen to be trivial.

5.2. **Hyperbolic localization on $\mathcal{M}_{0,m,G}$.** We recall some definitions and facts from [Ri19] in a form suited to our situation.

Definition 5.3. [Ri19, Definition 1.3] Let X/S be a morphism of schemes, with a $\mathbb{G}_m$-action on X/S (trivial on S). Define three functors on S-schemes (writing $X_T := X \times_S T$):

$$X^0 : T \mapsto \mathrm{Hom}_T^{\mathbb{G}_m}(T, X_T), \qquad X^+ : T \mapsto \mathrm{Hom}_T^{\mathbb{G}_m}((\mathbb{A}^1_T)^+, X_T), \qquad X^- : T \mapsto \mathrm{Hom}_T^{\mathbb{G}_m}((\mathbb{A}^1_T)^-, X_T),$$

where $\mathbb{G}_m$ acts on $(\mathbb{A}^1_T)^+$ by the usual multiplication, on $(\mathbb{A}^1_T)^-$ by its inverse, and trivially on T. The functor X^0 is the functor of fixed points; X^+ (resp. X^-) is the *attractor* (resp. *repeller*). There are natural morphisms [Ri19, Sec. 1.6]

$$\begin{tikzcd} & X^{\pm} \arrow[dl, "q^{\pm}"'] \arrow[dr, "p^{\pm}"] & \\ X^0 & & X, \end{tikzcd} \tag{5.2}$$

where $q^{\pm}$ (resp. $p^{\pm}$) is evaluation at the zero (resp. unit) section.

Remark 5.4. Formation of X^0, X^+, and X^- commutes with base change. For $X/\mathrm{Spec}(\mathbb{Z}_p)$ with a $\mathbb{G}_m$-action, we write X^0_η, X^+_η, X^-_η, etc., without ambiguity.

To ensure representability of these functors, we use the following notion. A $\mathbb{G}_m$-action on X/S is called *étale locally linearizable* if there exists a $\mathbb{G}_m$-equivariant étale covering family $\{U_i \to X\}_i$ with each U_i affine over S and carrying a $\mathbb{G}_m$-action.

Lemma 5.5. *Let X/S be a morphism between schemes with an étale locally linearizable $\mathbb{G}_m$-action, then X^0, X^+, and X^- are representable by schemes.*

Proof. This is a special case of [Ri19, Theorem A]. $\square$

If the covering family $\{U_i \to X\}_i$ can be chosen Zariski, we say the action is *Zariski locally linearizable*. The $\mathbb{G}_m$-action on $\mathcal{M}_{0,m,G}$ is of this form.

Lemma 5.6. *The $\mathbb{G}_m$-action on $\mathcal{M}_{0,m,G}$ is Zariski locally linearizable. Hence $\mathcal{M}^0_{0,m,G}$, $\mathcal{M}^+_{0,m,G}$, and $\mathcal{M}^-_{0,m,G}$ are representable by schemes.*

Proof. As in [HR21, Lemma 3.3], there is a natural $\mathbb{G}_m$-equivariant closed embedding of $\mathcal{M}_{0,m,G}$ into a product of classical Grassmannians, and the induced $\mathbb{G}_m$-action on the target (given by scalar multiplication on coordinates) is Zariski locally linearizable. This yields the claim. ☐

Remark 5.7. In general, for any $\mathbb{G}_m$-action on X/S with S a quasi-separated algebraic space and X a quasi-separated algebraic space locally of finite presentation over S, the action is étale locally linearizable [AHR, Theorem 10.1]. This applies to $\mathcal{M}_{0,m,G}/\mathrm{Spec}(\mathbb{Z}_p)$ since $\mathcal{M}_{0,m,G}$ is projective over $\mathrm{Spec}(\mathbb{Z}_p)$.

For our $\mathbb{G}_m$-action on $\mathcal{M}_{0,m,G}$, the "+" version of (5.2) reads

(5.3)
$$\begin{tikzcd} & \mathcal{M}^+_{0,m,G} \arrow[dl, "q^+"'] \arrow[dr, "p^+"] & \\ \mathcal{M}^0_{0,m,G} & & \mathcal{M}_{0,m,G}. \end{tikzcd}$$

5.2.1. *Hyperbolic localization on the generic fiber.* Over the generic point $\mathrm{Spec}(\mathbb{Q}_p)$, Diagram 5.1 is naturally isomorphic to Diagram 5.3, mirroring [HR21, Proposition 3.4]. We first record two lemmas.

Lemma 5.8. *Let R be a ring and $X \subset Y$ a closed embedding of R-schemes. If a morphism $f : \mathbb{A}^1_R \to Y$ restricts to a morphism $f|_{\mathbb{G}_{m,R}}$ factoring through X, then f itself factors through X.*

Proof. Let $I \subset R[t]$ be the ideal cutting out the preimage of X in $\mathbb{A}^1_R$. Our assumption implies $t^{-1}I = 0$. Since t is a non zero-divisor in $R[t]$, we have $I = 0$, hence f factors through X. ☐

Lemma 5.9. *Let R be a ring and $X \subset Y$ a $\mathbb{G}_m$-equivariant closed embedding of R-schemes with an étale locally linearizable $\mathbb{G}_m$-action on Y. Then the induced map $X^+ \to Y^+$ is a closed embedding and the diagram*

$$\begin{tikzcd} X^+ \arrow[r] \arrow[d] & Y^+ \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

is Cartesian.

Proof. For any $R \to R'$, a point of $(X \times_Y Y^+)(R')$ is a $\mathbb{G}_m$-equivariant map $g : (\mathbb{A}^1_{R'})^+ \to Y_{R'}$ with $g(1) \in X(R')$. Since X is $\mathbb{G}_m$-stable, $g|_{\mathbb{G}_{m,R'}}$ factors through $X_{R'}$, and by Lemma 5.8 so does g. Hence $(X \times_Y Y^+)(R') = X^+(R')$, proving the Cartesian statement. The morphism $X^+ \to Y^+$ is a closed embedding because $X \to Y$ is. ☐

Remark 5.10. The closed embedding $X^+ \to Y^+$ in Lemma 5.9 also appears under different hypotheses in [HR21, Corollary 2.3].

Proposition 5.11. *Over $\mathrm{Spec}(\mathbb{Q}_p)$, there is a natural isomorphism between Diagram 5.1 and Diagram 5.3:*

$$\begin{tikzcd} \mathcal{M}_{0,m,T,\eta} \arrow[d] & \mathcal{M}_{0,m,B,\eta} \arrow[l] \arrow[r] \arrow[d] & \mathcal{M}_{0,m,G,\eta} \arrow[d, "="'] \\ \mathcal{M}^0_{0,m,G,\eta} & \mathcal{M}^+_{0,m,G,\eta} \arrow[l, "q^+"'] \arrow[r, "p^+"] & \mathcal{M}_{0,m,G,\eta}. \end{tikzcd}$$

Proof. There is a $\mathbb{G}_m$-action on Gr_G as well: for a $\mathbb{Q}_p$-algebra R, elements of $T(R)$ act on $R(\!(t)\!)^n$ by coordinatewise scalar multiplication, hence on $R[\![t]\!]$-lattices; composing with $\lambda : \mathbb{G}_m \to T$ gives a $\mathbb{G}_m$-action on Gr_G. With this action, the embedding of Lemma 4.4 is $\mathbb{G}_m$-equivariant and identifies $\mathcal{M}_{0,m,G,\eta}$ with the locus of $R[\![t]\!]$-lattices in $R(\!(t)\!)^n$ bounded by m. This matches [HR21, Sec. 3.3] (their cocharacter χ plays the role of our λ).

First, $\mathcal{M}^0_{0,m,G,\eta} = \mathcal{M}_{0,m,T,\eta}$. This follows from the Cartesian diagram of Lemma 4.9

$$\begin{tikzcd} \mathcal{M}_{0,m,T,\eta} \arrow[r] \arrow[d] & \mathrm{Gr}_T \arrow[d] \\ \mathcal{M}_{0,m,G,\eta} \arrow[r] & \mathrm{Gr}_G \end{tikzcd}$$

together with $\mathrm{Gr}^0_G = \mathrm{Gr}_T$ [HR21, Proposition 3.4]. Concretely, $\mathcal{M}^0_{0,m,G,\eta}$ consists of $\mathbb{G}_m$-fixed lattices bounded by m, i.e. precisely the image of $\mathcal{M}_{0,m,T,\eta}$.

Next, $\mathcal{M}^+_{0,m,G,\eta} = \mathcal{M}_{0,m,B,\eta}$. By [HR21, Proposition 3.4], we have $\mathrm{Gr}^+_G = \mathrm{Gr}_B \subset \mathrm{Gr}_G$. The diagram

$$\begin{tikzcd} \mathcal{M}^+_{0,m,G,\eta} \arrow[r] \arrow[d] & \mathrm{Gr}^+_G \arrow[d] \\ \mathcal{M}_{0,m,G,\eta} \arrow[r] & \mathrm{Gr}_G \end{tikzcd}$$

is Cartesian by Lemma 5.9, since $\mathcal{M}_{0,m,G,\eta} \subset \mathrm{Gr}_G$ is a $\mathbb{G}_m$-equivariant closed embedding. The claim then follows from Lemma 4.21, which asserts that

$$\begin{tikzcd} \mathcal{M}_{0,m,B,\eta} \arrow[r] \arrow[d] & \mathrm{Gr}_B \arrow[d] \\ \mathcal{M}_{0,m,G,\eta} \arrow[r] & \mathrm{Gr}_G \end{tikzcd}$$

is Cartesian. Under these identifications, p^+ corresponds to the inclusion $\mathcal{M}_{0,m,B,\eta} \subset \mathcal{M}_{0,m,G,\eta}$.

Finally, q^+ corresponds to $\mathcal{M}_{0,m,B,\eta} \to \mathcal{M}_{0,m,T,\eta}$. Indeed, the $\mathbb{G}_m$-action on $\mathcal{M}_{0,m,B}$ leaves the consecutive quotients $\mathcal{Q}_i$ unchanged by definition; using that sections of $\mathcal{M}_{0,m,G,R} \to \mathrm{Spec}(R)$ are closed embeddings (since $\mathcal{M}_{0,m,G} \to \mathrm{Spec}(\mathbb{Z}_p)$ is separated) and Lemma 5.8, we obtain the identification. □

5.2.2. *Hyperbolic localization on the special fiber.* The special fiber of (5.3),

$$\begin{tikzcd} & \mathcal{M}^+_{0,m,G,s} \arrow[dl, "q^+"'] \arrow[dr, "p^+"] & \\ \mathcal{M}^0_{0,m,G,s} & & \mathcal{M}_{0,m,G,s}, \end{tikzcd} \tag{5.4}$$

is subtler: in general the equalities $\mathcal{M}^0_{0,m,G,s} = \mathcal{M}_{0,m,T,s}$ and $\mathcal{M}^+_{0,m,G,s} = \mathcal{M}_{0,m,B,s}$ do **not** hold. Nevertheless, for $G = \mathrm{GL}_n$ (and similarly GSp_{2g}), the scheme $\mathcal{M}^0_{0,m,G,s}$ (resp. $\mathcal{M}^+_{0,m,G,s}$) contains $\mathcal{M}_{0,m,T,s}$ (resp. $\mathcal{M}_{0,m,B,s}$) as an open and closed subscheme, with $(q^+)^{-1}(\mathcal{M}_{0,m,T,s}) = \mathcal{M}_{0,m,B,s}$.

Example 5.12. Consider $G = \mathrm{GL}_2$. Points of Fl_G are t-periodic lattice chains $\mathfrak{L}_\bullet : \mathfrak{L}_0 \subset \mathfrak{L}_1 \subset \mathfrak{L}_2 = t^{-1}\mathfrak{L}_0$. The $\mathbb{G}_m$-action is induced by the T-action on coordinates via λ and translates such chains.

If $\mathfrak{L}_\bullet$ is $\mathbb{G}_m$-fixed, then $\mathfrak{L}_0$ and $\mathfrak{L}_1$ are fixed. Using $\mathrm{Gr}^0_G = \mathrm{Gr}_T$ [HR21, Proposition 3.4], we may write

$$\mathfrak{L}_\bullet : \quad \mathcal{U}_1 \oplus \mathcal{U}_2 \subset \mathcal{V}_1 \oplus \mathcal{V}_2 \subset t^{-1}\mathcal{U}_1 \oplus t^{-1}\mathcal{U}_2,$$

with $\mathcal{U}_i$ (resp. $\mathcal{V}_i$) rank-1 lattices in the i-th coordinate. By rank considerations, the only possibilities are

$$\mathcal{V}_1 \oplus \mathcal{V}_2 = t^{-1}\mathcal{U}_1 \oplus \mathcal{U}_2 \quad \text{or} \quad \mathcal{U}_1 \oplus t^{-1}\mathcal{U}_2.$$

The embedding $\mathrm{Fl}_T \subset \mathrm{Fl}_G$ yields the first case; the second is obtained by left-multiplying by (the permutation matrix representing) the nontrivial element $a \in S_2$ (the Weyl group). Thus

$$\mathrm{Fl}_G^0 = \mathrm{Fl}_T \coprod a(\mathrm{Fl}_T).$$

If $\mathfrak{L}_\bullet \in \mathrm{Fl}_G^+$, then the natural $\mathbb{G}_m$-equivariant maps $\mathrm{Fl}_G \to \mathrm{Gr}_G$ sending $\mathfrak{L}_\bullet$ to $\mathfrak{L}_0$ and to $\mathfrak{L}_1$ show that $\mathfrak{L}_0, \mathfrak{L}_1 \in \mathrm{Gr}_B$. Writing $\mathfrak{L}_0$ (resp. $\mathfrak{L}_1$) as a chain $\mathcal{C}_\bullet$ (resp. $\mathcal{C}'_\bullet$) as in Lemma 4.20, one finds two disjoint cases by considering the image $q^+(\mathfrak{L}_\bullet)$; in the first, $\mathfrak{L}_1 = \mathfrak{L}_0 + t^{-1}\mathcal{C}_1$ and $\mathfrak{L}_\bullet$ corresponds to $\mathrm{Fl}_B \subset \mathrm{Fl}_G$, and in the second, $\mathfrak{L}_2 = \mathfrak{L}_1 + t^{-1}\mathcal{C}'_1$ and the shift of $\mathfrak{L}_\bullet$ corresponds to Fl_B. Hence

$$\mathrm{Fl}_G^+ = \mathrm{Fl}_B \coprod s(\mathrm{Fl}_B),$$

where $s : \mathfrak{L}_\bullet \mapsto \mathfrak{L}_{\bullet-1}$ is the shift automorphism on Fl_G. Moreover, $q^+(\mathrm{Fl}_B) = \mathrm{Fl}_T$ and $q^+(s(\mathrm{Fl}_B)) = a(\mathrm{Fl}_T)$, so $(q^+)^{-1}(\mathrm{Fl}_T) = \mathrm{Fl}_B$. By Lemma 5.9, the analogous statements hold for $\mathcal{M}_{0,m,G,s}$.

Proposition 5.13. *Over* $\mathrm{Spec}(\mathbb{F}_p)$*, there is a natural morphism between Diagram 5.1 and Diagram 5.4*

$$\begin{tikzcd}
\mathcal{M}_{0,m,T,s} \arrow[d, "\iota"] & \mathcal{M}_{0,m,B,s} \arrow[l] \arrow[r] \arrow[d, "\iota^+"] & \mathcal{M}_{0,m,G,s} \arrow[d, "="'] \\
\mathcal{M}^0_{0,m,G,s} & \mathcal{M}^+_{0,m,G,s} \arrow[l, "q^+"'] \arrow[r, "p^+"] & \mathcal{M}_{0,m,G,s}.
\end{tikzcd}$$

where ι and ι^+ are open and closed embeddings, and the left square is Cartesian.

Proof. The argument follows Example 5.12. For a $\mathbb{G}_m$-fixed t-periodic lattice chain

$$\mathfrak{L}_0 \subset \mathfrak{L}_1 \subset \cdots \subset \mathfrak{L}_n = t^{-1}\mathfrak{L}_0,$$

write $\mathfrak{L}_0 = \mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_n$ [HR21, Proposition 3.4]. At each step one replaces a single $\mathcal{U}_i$ by $t^{-1}\mathcal{U}_i$, yielding $n!$ possibilities, naturally indexed by the Weyl group W. Thus

$$\mathcal{M}^0_{0,m,G,s} = \coprod_{w \in W} w(\mathcal{M}_{0,m,T,s}),$$

where w acts via the corresponding permutation of coordinates.

For $\mathfrak{L}_\bullet$ in the attractor, there are again $n!$ candidates for $q^+(\mathfrak{L}_\bullet)$. By rank considerations, $q^+(\mathfrak{L}_\bullet) \in \mathcal{M}_{0,m,T,s} \subset \mathcal{M}_{0,m,G,s}$ if and only if $\mathfrak{L}_\bullet \in \mathcal{M}_{0,m,B,s}$. Hence $(q^+)^{-1}(\mathcal{M}_{0,m,T,s}) = \mathcal{M}_{0,m,B,s}$, and the left square is Cartesian. $\square$

Remark 5.14. Applying the shift automorphism of Fl_G (as in Example 5.12) to Proposition 5.13, one finds that for some (but not all) $w \neq 1$ in W (namely, those induced by shifts), the preimage $(q^+)^{-1}(w(\mathcal{M}_{0,m,T,s}))$ is a copy of $\mathcal{M}_{0,m,B,s}$.

Remark 5.15. Proposition 5.13 is an analogue of the "+"-diagram in [HR21, Proposition 4.7]. One can also obtain it by pulling back that diagram (note that $\mathcal{P}^+$ there is B under our assumptions) and applying Lemma 5.9. Since good moduli interpretations are available in our setting, we chose a more direct proof.

6. Truncated deformations for the $\Gamma_1(p)$-level

So far we have worked at the Iwahori level. We now pass to the $\Gamma_1(p)$-level.

First, we introduce the geometric objects that are T-torsors over $\mathcal{M}_{0,m,G}$, $\mathcal{M}_{0,m,T}$, and $\mathcal{M}_{0,m,B}$. Note that the notations for these objects in this paper are slightly different from those in [HLS]. In [HLS], the T-torsor over $\mathcal{M}_{0,m,G}$ is denoted by $\mathcal{M}^+_{0,m,G}$, but since we already use this notation for the attractor, we will instead write $\mathcal{M}^t_{0,m,G}$ for the T-torsor over $\mathcal{M}_{0,m,G}$.

We briefly review determinant line bundles and their distinguished sections attached to complexes. For details, see [HLS, Sec.2], [KM], and [Knud].

Let $C = [V \xrightarrow{\alpha} W]$ be a complex of finite-rank projective $\mathcal{O}_S$-modules with $\mathrm{rank}(V) = \mathrm{rank}(W)$ (i.e. C has virtual rank 0). The determinant line bundle $\mathrm{Det}(C)$ attached to C is defined as $\mathrm{Det}(W)\otimes$

$\mathrm{Det}(V)^{-1}$ (for a finitely generated projective module, Det denotes the top exterior power), and it is equipped with the distinguished section

$$\mathcal{O}_S \cong \mathrm{Det}(V) \otimes \mathrm{Det}(V)^{-1} \xrightarrow{\mathrm{Det}(\alpha)\otimes\mathrm{id}} \mathrm{Det}(W) \otimes \mathrm{Det}(V)^{-1} \;=:\; \mathrm{Det}(C).$$

Recall the following definition from [HLS, Sec.7.3] (there stated for GSp_{2g}; we adapt it to GL_n).

Definition 6.1. Let $\mathcal{M}^t_{0,m,G}$ be the T-torsor over $\mathcal{M}_{0,m,G}$ parametrizing isomorphisms

$$\varphi_i : \ \mathrm{Det}\Big[\frac{\mathcal{W}_{i-1}}{t^{m^+}\mathbb{V}_{i-1}[t]_{\mathcal{O}_S}} \longrightarrow \frac{\mathcal{W}_i}{t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S}}\Big] \;\cong\; \mathcal{O}_S.$$

We define $\mathcal{M}^t_{0,m,T}$ and $\mathcal{M}^t_{0,m,B}$ by pulling back this T-torsor along the embeddings $\mathcal{M}_{0,m,T} \subset \mathcal{M}_{0,m,B} \subset \mathcal{M}_{0,m,G}$ in Lemma 4.17, obtaining embeddings

$$\mathcal{M}^t_{0,m,T} \;\subset\; \mathcal{M}^t_{0,m,B} \;\subset\; \mathcal{M}^t_{0,m,G}.$$

The scheme $\mathcal{M}^t_{0,m,G}$ serves as a truncated deformation to a T-torsor over the affine grassmannian from the enhanced affine flag variety Fl^t_G (Definition 3.6). We define Fl^t_T and Fl^t_B by pulling back $\mathrm{Fl}^t_G \to \mathrm{Fl}_G$ (Definition 3.6) along $\mathrm{Fl}_T \subset \mathrm{Fl}_B \subset \mathrm{Fl}_G$.

Lemma 6.2. *There exists a canonical closed embedding of the special fiber $\mathcal{M}^t_{0,m,G,s}$ of $\mathcal{M}^t_{0,m,G}$ into Fl^t_G lifting the closed embedding in Lemma 4.6, and the resulting diagram*

$$\begin{tikzcd} \mathcal{M}^t_{0,m,G,s} \arrow[r] \arrow[d] & \mathrm{Fl}^t_G \arrow[d] \\ \mathcal{M}_{0,m,G,s} \arrow[r] & \mathrm{Fl}_G \end{tikzcd}$$

is Cartesian. Analogous statements hold with G replaced by T and by B.

Proof. For G, this is [HLS, Lemma 9.3.2]. The key point is the canonical isomorphism ([HLS, Lemma 2.3.2])

$$\mathrm{Det}\Big(\Big[\mathcal{W}_{i-1}/t^{m^+}\mathbb{V}_{i-1}[t]_{\mathcal{O}_S} \;\longrightarrow\; \mathcal{W}_i/t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S}\Big]\Big) \;=\; (\mathcal{W}_i/\mathcal{W}_{i-1}) \otimes (\mathbb{V}_i[t]_{\mathcal{O}_S}/\mathbb{V}_{i-1}[t]_{\mathcal{O}_S})^{-1},$$

and the factor $\big(\mathbb{V}_i[t]/\mathbb{V}_{i-1}[t]\big)$ admits a canonical trivialization using the basis $e_i \in \mathbb{V}$. Thus the consecutive determinant line bundles identify with the consecutive quotients.

The cases of T and B follow by base change. □

7. The $\mathbb{G}_m$-action and hyperbolic localization for the $\Gamma_1(p)$-level

In this section, we generalize the results of §5 to the setting of T-torsors.

First, we specify the analogue of Diagram 5.1 for $\Gamma_1(p)$-level objects. The morphism $\mathcal{M}^t_{0,m,B} \subset \mathcal{M}^t_{0,m,G}$ is given in Definition 6.1, so it remains to define a morphism $\mathcal{M}^t_{0,m,B} \to \mathcal{M}^t_{0,m,T}$.

Lemma 7.1. *There exists a natural morphism $\mathcal{M}^t_{0,m,B} \to \mathcal{M}^t_{0,m,T}$ lifting $\mathcal{M}_{0,m,B} \to \mathcal{M}_{0,m,T}$, and the resulting diagram*

$$\begin{tikzcd} \mathcal{M}^t_{0,m,B} \arrow[r] \arrow[d] & \mathcal{M}^t_{0,m,T} \arrow[d] \\ \mathcal{M}_{0,m,B} \arrow[r] & \mathcal{M}_{0,m,T} \end{tikzcd}$$

is Cartesian.

Proof. We need a natural isomorphism

$$\mathrm{Det}\Big[\frac{\mathcal{W}_{i-1}}{t^{m^+}\mathbb{V}_{i-1}[t]_{\mathcal{O}_S}} \to \frac{\mathcal{W}_i}{t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S}}\Big] \;\cong\; \mathrm{Det}\Big[\frac{\mathcal{Q}_i}{t^{m^+}\mathcal{O}_S[t]} \to \frac{(t+w_p)^{-1}\mathcal{Q}_i}{(t+w_p)^{-1}t^{m^+}\mathcal{O}_S[t]}\Big].$$

By [HLS, Lemma 2.3.2] there are isomorphisms of determinant line bundles

$$\mathrm{Det}\Big[\frac{\mathcal{W}_{i-1}}{t^{m^+}\mathbb{V}_{i-1}[t]_{\mathcal{O}_S}} \to \frac{\mathcal{W}_i}{t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S}}\Big] \;\cong\; \mathrm{Det}\Big[\frac{\mathcal{W}_i}{\mathcal{W}_{i-1}} \to \frac{t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S}}{t^{m^+}\mathbb{V}_{i-1}[t]_{\mathcal{O}_S}}\Big],$$

$$\mathrm{Det}\Big[\frac{\mathcal{Q}_i}{t^{m^+}\mathcal{O}_S[t]} \to \frac{(t+w_p)^{-1}\mathcal{Q}_i}{(t+w_p)^{-1}t^{m^+}\mathcal{O}_S[t]}\Big] \cong \mathrm{Det}\Big[\frac{(t+w_p)^{-1}\mathcal{Q}_i}{\mathcal{Q}_i} \to \frac{(t+w_p)^{-1}t^{m^+}\mathcal{O}_S[t]}{t^{m^+}\mathcal{O}_S[t]}\Big].$$

From the proof of Lemma 4.17, where it is shown that $\frac{\mathcal{W}_i}{\mathcal{W}_{i-1}} = \frac{(t+w_p)^{-1}\mathcal{Q}_i}{\mathcal{Q}_i}$, we obtain an isomorphism of complexes

$$\Big[\frac{\mathcal{W}_i}{\mathcal{W}_{i-1}} \to \frac{t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S}}{t^{m^+}\mathbb{V}_{i-1}[t]_{\mathcal{O}_S}}\Big] \cong \Big[\frac{(t+w_p)^{-1}\mathcal{Q}_i}{\mathcal{Q}_i} \to \frac{(t+w_p)^{-1}t^{m^+}\mathcal{O}_S[t]}{t^{m^+}\mathcal{O}_S[t]}\Big].$$

Applying the determinant functor yields the desired natural isomorphism. Since the isomorphisms in [HLS, Lemma 2.3.2] preserve the distinguished sections, and any isomorphism of complexes preserves distinguished sections under the determinant, the natural isomorphism above identifies the distinguished sections. □

Thus we obtain the diagram of T-torsors

(7.1)
$$\begin{tikzcd} & \mathcal{M}^t_{0,m,B} \arrow[dl] \arrow[dr] & \\ \mathcal{M}^t_{0,m,T} & & \mathcal{M}^t_{0,m,G}. \end{tikzcd}$$

Lemma 7.2. *The two squares formed by Diagram 5.1 and Diagram 7.1*

$$\begin{tikzcd} \mathcal{M}^t_{0,m,T} \arrow[d] & \mathcal{M}^t_{0,m,B} \arrow[l] \arrow[r] \arrow[d] & \mathcal{M}^t_{0,m,G} \arrow[d] \\ \mathcal{M}_{0,m,T} & \mathcal{M}_{0,m,B} \arrow[l] \arrow[r] & \mathcal{M}_{0,m,G}. \end{tikzcd}$$

are Cartesian.

Proof. This follows from Definition 6.1 and Lemma 7.1. □

The action of I_m on $\mathcal{M}_{0,m,G}$ (§5.1) lifts naturally to an action on $\mathcal{M}^t_{0,m,G}$ by translating trivializations ([HLS, Lemma 7.5.1]). Hence we obtain a $\mathbb{G}_m$-action on $\mathcal{M}^t_{0,m,G}$ such that the projection $\mathcal{M}^t_{0,m,G} \to \mathcal{M}_{0,m,G}$ is $\mathbb{G}_m$-equivariant. Since $\mathcal{M}^t_{0,m,G} \to \mathcal{M}_{0,m,G}$ is affine and the $\mathbb{G}_m$-action on $\mathcal{M}_{0,m,G}$ is Zariski locally linearizable (Lemma 5.6), the $\mathbb{G}_m$-action on $\mathcal{M}^t_{0,m,G}$ is also Zariski locally linearizable. Therefore, the objects obtained by hyperbolic localization on $\mathcal{M}^t_{0,m,G}$ with respect to this $\mathbb{G}_m$-action are representable by schemes (Lemma 5.5). We have the following diagram of schemes:

(7.2)
$$\begin{tikzcd} & \mathcal{M}^{t,+}_{0,m,G} \arrow[dl] \arrow[dr] & \\ \mathcal{M}^{t,0}_{0,m,G} & & \mathcal{M}^t_{0,m,G}. \end{tikzcd}$$

Remark 7.3. When we mention the T-action on $\mathcal{M}^t_{0,m,G}$, $\mathcal{M}^t_{0,m,T}$, and $\mathcal{M}^t_{0,m,B}$, we always mean the T-action on the trivializations in Definition 6.1, rather than the T-action induced by the I_m-action. For the latter, we only use the induced $\mathbb{G}_m$-action to avoid ambiguity. By definition, the I_m-action (and hence the $\mathbb{G}_m$-action) commutes with the T-action.

We now show that over the generic point $\mathrm{Spec}(\mathbb{Q}_p)$, there is a natural isomorphism between Diagrams 7.1 and 7.2, generalizing Proposition 5.11 to T-torsors. Before proving this, we observe that on the generic fiber the T-torsor $\mathcal{M}^t_{0,m,G,\eta} \to \mathcal{M}_{0,m,G,\eta}$ is actually trivial.

Lemma 7.4. *The T-torsor $\mathcal{M}^t_{0,m,G,\eta} \to \mathcal{M}_{0,m,G,\eta}$ is trivial. Moreover, there exists an $I_{m,\eta}$-equivariant (hence $\mathbb{G}_{m,\eta}$-equivariant) canonical section $\mathrm{can}_{1,m}$ trivializing this T-torsor. When G is replaced by T or by B, analogous statements hold. We abuse notation and denote all these canonical sections by $\mathrm{can}_{1,m}$.*

Proof. It suffices to treat G, since the other cases follow by base change. To define $\mathrm{can}_{1,m}$, we specify the canonical trivializations φ_i in Definition 6.1 for any $\mathcal{W}_i$, i.e. an invertible section of

$$\mathrm{Det}\Big[\frac{\mathcal{W}_{i-1}}{t^{m^+}\mathbb{V}_{i-1}[t]_{\mathcal{O}_S}} \to \frac{\mathcal{W}_i}{t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S}}\Big].$$

We claim that the distinguished section of this determinant line bundle is invertible on the generic fiber. Indeed, the morphism $\frac{\mathcal{W}_{i-1}}{t^{m^+}\mathbb{V}_{i-1}[t]_{\mathcal{O}_S}} \to \frac{\mathcal{W}_i}{t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S}}$ is an isomorphism over the generic fiber by Lemma 4.3. Thus the distinguished section provides a section of the T-torsor. This section is I_m-equivariant, since the I_m-action on $\mathcal{M}_{0,m,G}$ preserves distinguished sections (it induces isomorphisms of the underlying complexes). □

Proposition 7.5. *Over* $\mathrm{Spec}(\mathbb{Q}_p)$*, there is a natural isomorphism between Diagram 7.1 and Diagram 7.2*

$$\begin{tikzcd}
\mathcal{M}^t_{0,m,T,\eta} \arrow[d] & \mathcal{M}^t_{0,m,B,\eta} \arrow[l] \arrow[r] \arrow[d] & \mathcal{M}^t_{0,m,G,\eta} \arrow[d, "="'] \\
\mathcal{M}^{t,0}_{0,m,G,\eta} & \mathcal{M}^{t,+}_{0,m,G,\eta} \arrow[l, "q^{t,+}"'] \arrow[r, "p^{t,+}"] & \mathcal{M}^t_{0,m,G,\eta},
\end{tikzcd}$$

and this isomorphism lifts that of Proposition 5.11.

Proof. By Lemma 7.4, we have an identification $\mathcal{M}^t_{0,m,G,\eta} = \mathcal{M}_{0,m,G,\eta} \times T$, with $\mathbb{G}_m$ acting trivially on the T-factor. Hence

$$\mathcal{M}^{t,0}_{0,m,G,\eta} = \mathcal{M}^0_{0,m,G,\eta} \times T = \mathcal{M}_{0,m,T,\eta} \times T = \mathcal{M}^t_{0,m,T,\eta},$$

$$\mathcal{M}^{t,+}_{0,m,G,\eta} = \mathcal{M}^+_{0,m,G,\eta} \times T = \mathcal{M}_{0,m,B,\eta} \times T = \mathcal{M}^t_{0,m,B,\eta},$$

as required. □

Over the special point $\mathrm{Spec}(\mathbb{F}_p)$, Diagram 7.1 becomes even simpler for the $\Gamma_1(p)$-level. Recall that there is a natural closed embedding $\mathcal{M}^t_{0,m,G,s} \subset \mathrm{Fl}^t_G$ (Lemma 6.2). Define a T-action on Fl^t_G by

$$(t_\bullet) \cdot (\mathfrak{L}_\bullet, \phi_\bullet) := (\mathfrak{L}'_\bullet, \phi'_\bullet),$$

where $\mathfrak{L}'_i := t_\bullet \mathfrak{L}_i$ (translation of the t-periodic lattice chain) and

$$\phi'_i : \ \mathfrak{L}'_i/\mathfrak{L}'_{i-1} \xrightarrow{t_\bullet^{-1}} \mathfrak{L}_i/\mathfrak{L}_{i-1} \xrightarrow{\phi_i} \mathcal{O}_S \xrightarrow{\times t_i} \mathcal{O}_S.$$

Using the cocharacter λ, we obtain a $\mathbb{G}_m$-action on Fl^t_G. Both $\mathrm{Fl}^t_G \to \mathrm{Fl}_G$ and $\mathcal{M}^t_{0,m,G,s} \subset \mathrm{Fl}^t_G$ are $\mathbb{G}_m$-equivariant ([HLS, Lemma 9.3.2]).

Remark 7.6. In the definition of ϕ'_i, the final factor $\mathcal{O}_S \xrightarrow{\times t_i} \mathcal{O}_S$ is essential; without it, even Fl^t_T would not be fixed by the $\mathbb{G}_m$-action.

Proposition 7.7. *Over the special point* $\mathrm{Spec}(\mathbb{F}_p)$*, there exists a natural isomorphism between Diagram 7.1 and Diagram 7.2*

$$\begin{tikzcd}
\mathcal{M}^t_{0,m,T,s} \arrow[d] & \mathcal{M}^t_{0,m,B,s} \arrow[l] \arrow[r] \arrow[d] & \mathcal{M}^t_{0,m,G,s} \arrow[d, "="'] \\
\mathcal{M}^{t,0}_{0,m,G,s} & \mathcal{M}^{t,+}_{0,m,G,s} \arrow[l, "q^{t,+}"'] \arrow[r, "p^{t,+}"] & \mathcal{M}^t_{0,m,G,s},
\end{tikzcd}$$

and this isomorphism lifts that of Proposition 5.13.

Proof. Let $f : \mathcal{M}^t_{0,m,G} \to \mathcal{M}_{0,m,G}$ be the $\mathbb{G}_m$-equivariant projection. Then $f(\mathcal{M}^{t,0}_{0,m,G,s}) \subset \mathcal{M}^0_{0,m,G,s}$. By the proof of Proposition 5.13, $\mathcal{M}^0_{0,m,G,s} = \coprod_{w\in W} w(\mathcal{M}_{0,m,T,s})$. For $x \in f^{-1}(\mathcal{M}^0_{0,m,G,s})$, we have $x \in \mathcal{M}^{t,0}_{0,m,G,s}$ if and only if $f(x)$ lies in the component corresponding to $1 \in W$. Indeed, on the component indexed by $w \in W$, the element $t \in \mathbb{G}_m$ acts by ${}^w\lambda(t)^{-1} \cdot \lambda(t)$ (with ${}^w\lambda(t) := \lambda(w^{-1}tw)$) via the T-action as a T-torsor, and this equals 1 for all t if and only if $w = 1$ (our fixed λ is regular).

Similarly, $f(\mathcal{M}^{t,+}_{0,m,G,s}) \subset \mathcal{M}^+_{0,m,G,s}$. From the preceding analysis, $f(\mathcal{M}^{t,+}_{0,m,G,s})$ lies in $\mathcal{M}_{0,m,B,s} \subset \mathcal{M}^+_{0,m,G,s}$, since the other components disappear. Note that the $\mathbb{G}_m$-action on $\mathcal{M}_{0,m,B,s}$ does not change the consecutive quotients $\mathcal{Q}_i$; by Lemma 7.1, the full preimage $f^{-1}(\mathcal{M}_{0,m,B,s})$ lies in the attractor and hence equals $\mathcal{M}^{t,+}_{0,m,G,s}$. □

8. Sheaves on $\mathcal{M}_{0,m,G,\eta}$ and $\mathcal{M}^t_{0,m,G,\eta}$

In this section, we define certain (shifted) perverse sheaves on $\mathcal{M}_{0,m,G,\eta}$ and $\mathcal{M}^t_{0,m,G,\eta}$. As we will prove later, via the closed embeddings $\mathcal{M}_{0,m,G,s} \subset \mathrm{Fl}_G$ (Lemma 4.6) and $\mathcal{M}^t_{0,m,G,s} \subset \mathrm{Fl}^t_G$ (Lemma 6.2), the nearby cycles of these perverse sheaves yield central elements in the corresponding Hecke algebras and these functions can be described explicitly (Theorem 10.12).

8.1. **Construction of sheaves.** To obtain the desired sheaves on $\mathcal{M}^t_{0,m,G,\eta}$, we introduce geometric objects motivated by Oort–Tate theory [OT]. For more details, see [HLS, Sec. 2, 5, 7]. In what follows, a $(p-1)$-st root a' of some $a \in \mathcal{O}_S(S)$ means an element $a' \in \mathcal{O}_S(S)$ such that $(a')^{p-1} = a$.

Definition 8.1. Let $\mathcal{M}^t_{1,m,G}$ be the scheme over $\mathcal{M}^t_{0,m,G}$ parametrizing $(p-1)$-st roots of $\varphi_i(a_i)$, where
$$\varphi_i : \ \mathrm{Det}\Big[\frac{\mathcal{W}_{i-1}}{t^{m^+}\mathbb{V}_{i-1}[t]_{\mathcal{O}_S}} \to \frac{\mathcal{W}_i}{t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S}}\Big] \cong \mathcal{O}_S$$
is the trivialization of the determinant line bundle (Definition 6.1) and a_i is the distinguished section attached to this determinant line bundle. Acting on the trivial line bundle $\mathcal{O}_S$ on the right-hand side of φ_i defines a T-action on $\mathcal{M}^t_{1,m,G}$. Note that the morphism $\mathcal{M}^t_{1,m,G} \to \mathcal{M}^t_{0,m,G}$ is **not** T-equivariant; rather, it is equivariant with respect to $(p-1) : T \to T$, given by $t \mapsto t^{p-1}$. We define $\mathcal{M}^t_{1,m,T}$ and $\mathcal{M}^t_{1,m,B}$ by pulling back $\mathcal{M}^t_{1,m,G} \to \mathcal{M}^t_{0,m,G}$ along the embeddings $\mathcal{M}^t_{0,m,T} \subset \mathcal{M}^t_{0,m,B} \subset \mathcal{M}^t_{0,m,G}$ from Definition 6.1.

By definition, we obtain the diagram

$$\begin{tikzcd} & \mathcal{M}^t_{1,m,B} \arrow[dl] \arrow[dr] & \\ \mathcal{M}^t_{1,m,T} & & \mathcal{M}^t_{1,m,G} \end{tikzcd} \tag{8.1}$$

in which the arrow on the left is defined using Lemma 7.1. This diagram is compatible with Diagram 7.1.

Lemma 8.2. *The two squares formed by Diagram 7.1 and Diagram 8.1*

$$\begin{tikzcd} \mathcal{M}^t_{1,m,T} \arrow[d] & \mathcal{M}^t_{1,m,B} \arrow[l] \arrow[r] \arrow[d] & \mathcal{M}^t_{1,m,G} \arrow[d] \\ \mathcal{M}^t_{0,m,T} & \mathcal{M}^t_{0,m,B} \arrow[l] \arrow[r] & \mathcal{M}^t_{0,m,G}. \end{tikzcd}$$

are Cartesian.

Proof. This follows from Definition 8.1 and Lemma 7.1. □

By definition, the morphism $\mathcal{M}^t_{1,m,G} \to \mathcal{M}^t_{0,m,G}$ is finite. By Lemma 4.3, over the generic point $\mathrm{Spec}(\mathbb{Q}_p)$ the distinguished section a_i is invertible, so $\mathcal{M}^t_{1,m,G,\eta} \to \mathcal{M}^t_{0,m,G,\eta}$ is a finite étale cover with covering group the $(p-1)$-st roots of the identity in $T(\mathbb{Q}_p)$, which we identify with $T(\mathbb{F}_p)$ via Teichmüller lifting as mentioned in §2.

The morphism $\mathcal{M}^t_{1,m,G,\eta} \to \mathcal{M}^t_{0,m,G,\eta}$ admits a more explicit description.

Lemma 8.3. *There is a canonical isomorphism $\mathcal{M}^t_{1,m,G,\eta} \cong \mathcal{M}_{0,m,G,\eta} \times T$ such that, combined with the canonical trivialization $\mathcal{M}^t_{0,m,G,\eta} = \mathcal{M}_{0,m,G,\eta} \times T$ from Lemma 7.4, the morphism $\mathcal{M}^t_{1,m,G,\eta} \to \mathcal{M}^t_{0,m,G,\eta}$ is identified with*

$$\mathrm{id} \times (p-1): \ \mathcal{M}_{0,m,G,\eta} \times T \longrightarrow \mathcal{M}_{0,m,G,\eta} \times T,$$

where id *is the identity on $\mathcal{M}_{0,m,G,\eta}$ and $(p-1): T \to T$ is $t \mapsto t^{p-1}$, as in Definition 8.1.*

Proof. From the proof of Lemma 7.4, the trivialization φ_i defining the canonical section $\mathrm{can}_{1,m}$ sends a_i to $\varphi_i(a_i) = 1 \in \mathcal{O}_S(S)$. Since 1 is a $(p-1)$-st root of itself, we obtain a section $\mathcal{M}_{0,m,G,\eta} \to \mathcal{M}^t_{1,m,G,\eta}$, which gives the canonical isomorphism. $\square$

Remark 8.4. There is a natural I_m-action on $\mathcal{M}^t_{1,m,G}$ lifting the I_m-action on $\mathcal{M}^t_{0,m,G}$ [HLS, Lemma 7.5.5]. Using the fixed cocharacter λ, we obtain a $\mathbb{G}_m$-action on $\mathcal{M}^t_{1,m,G}$, which is again Zariski locally linearizable since $\mathcal{M}^t_{1,m,G} \to \mathcal{M}^t_{0,m,G}$ is $\mathbb{G}_m$-equivariant and affine. It therefore makes sense to consider hyperbolic localization on $\mathcal{M}^t_{1,m,G}$, and one can prove analogues of Proposition 7.5 and Proposition 7.7 for $\mathcal{M}^t_{1,m,G}$ using Lemma 8.3 (note that $\mathbb{G}_m$ acts only on the first factor). Since we do not need these results, we omit the proofs.

We now introduce sheaves on $\mathcal{M}_{0,m,G,\eta}$ and $\mathcal{M}^t_{0,m,G,\eta}$. Fix a prime $\ell \neq p$. All sheaves are $\overline{\mathbb{Q}}_\ell$-étale sheaves. As in [HR21], for a separated scheme X of finite type over a field F whose cyclotomic character $\mathrm{Gal}(\overline{F}/F) \to \mathbb{Z}_\ell^\times$, composed with $\mathbb{Z}_\ell^\times \hookrightarrow \overline{\mathbb{Z}}_\ell^\times$, admits a square root, we write $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ (or simply $D^b_c(X)$) for the bounded derived category of $\overline{\mathbb{Q}}_\ell$-complexes with constructible cohomology sheaves on X. We denote by $\mathrm{Perv}(X)$ the heart of the perverse t-structure.

For any $\overline{\mathbb{Q}}_\ell$-complex $\mathcal{A}$ and any integer $n \in \mathbb{Z}$, define

$$\mathcal{A}\langle n \rangle := \mathcal{A}[n](n/2), \tag{8.2}$$

where $(1/2)$ denotes the half Tate twist using the chosen square root of the cyclotomic character, and $[\cdot]$ is the usual cohomological shift.

Fix $\mu \in X(T)^+$ dominant with respect to B. Denote by $\mu(t)$ the image of $t \in \mathbb{G}_m(\mathbb{Q}_p(\!(t)\!))$ under

$$\mathbb{G}_m(\mathbb{Q}_p(\!(t)\!)) \xrightarrow{\mu} T(\mathbb{Q}_p(\!(t)\!)) \subset G(\mathbb{Q}_p(\!(t)\!)) \longrightarrow \mathrm{Gr}_G(\mathbb{Q}_p).$$

By enlarging m if necessary, we may (and do) assume $\mu(t)$ lies in the image of $\mathcal{M}_{0,m,G,\eta}$ under the embedding $\mathcal{M}_{0,m,G,\eta} \subset \mathrm{Gr}_G$ (Lemma 4.4).

Let $O_\mu \subset \mathcal{M}_{0,m,G,\eta}$ be the $I_{m,\eta}$-orbit (Definition 5.2) of $\mu(t)$, and let $\overline{O}_\mu$ be its closure in $\mathcal{M}_{0,m,G,\eta}$. For the inclusion $j: O_\mu \hookrightarrow \overline{O}_\mu$, define the normalized $\overline{\mathbb{Q}}_\ell$-intersection complex by middle extension

$$\mathcal{A}_\mu := j_{!*}\overline{\mathbb{Q}}_\ell\langle d_\mu \rangle,$$

where $d_\mu = \dim(O_\mu)$. Via the closed embedding $\overline{O}_\mu \subset \mathcal{M}_{0,m,G,\eta}$, the complex $\mathcal{A}_\mu$ is an $I_{m,\eta}$-equivariant perverse sheaf on $\mathcal{M}_{0,m,G,\eta}$. We may also regard $\mathcal{A}_\mu$ as a complex on Gr_G via the closed embedding $\mathcal{M}_{0,m,G,\eta} \subset \mathrm{Gr}_G$.

Remark 8.5. As a complex on Gr_G, $\mathcal{A}_\mu$ coincides with the usual perverse sheaf (e.g. $\mathrm{IC}_{\{\mu\}}$ in [HR21, (3.21)]) constructed using the Schubert stratification attached to μ. Indeed, the $I_{m,\eta}$-orbit O_μ, as a set, agrees with the L^+G-orbit of $\mu(t)$ in Gr_G, where $L^+G(R) := G(R[\![t]\!])$ is the usual positive loop group. Let k be a $\mathbb{Q}_p$-field. By the Chinese Remainder Theorem, $I_{m,\eta}(k)$ is isomorphic to the product of $\mathrm{GL}\big(\frac{\mathbb{V}_0[t]_k}{t^{m^+-m^-}\mathbb{V}_0[t]_k}\big)$ and the standard Borel subgroup of $\mathrm{GL}\big(\frac{\mathbb{V}_0[t]_k}{(t+w_p)\mathbb{V}_0[t]_k}\big)$ [HLS, Lemma 6.4.2]. The second factor does not change $\mathcal{W}_0$, and by Lemma 4.3 we only need the first factor for $I_{m,\eta}$-orbits. Since G is smooth over $\mathrm{Spec}(\mathbb{Z}_p)$, the reduction map

$$\mathrm{GL}\big(k[\![t]\!]^n\big) \longrightarrow \mathrm{GL}\big(\tfrac{k[\![t]\!]^n}{t^{m^+-m^-}k[\![t]\!]^n}\big)$$

is surjective, so $I_{m,\eta}$-orbits agree with L^+G-orbits.

To produce sheaves on $\mathcal{M}^t_{0,m,G,\eta}$, consider the I_m-equivariant morphisms (see Remark 8.4)

$$\mathcal{M}^t_{1,m,G} \xrightarrow{\pi} \mathcal{M}^t_{0,m,G} \longrightarrow \mathcal{M}_{0,m,G}.$$

Let $\mathcal{B}^+_\mu$ be the (shifted perverse) sheaf on $\mathcal{M}^t_{0,m,G,\eta}$ obtained by pulling back $\mathcal{A}_\mu$ along the T-torsor $\mathcal{M}^t_{0,m,G,\eta} \to \mathcal{M}_{0,m,G,\eta}$. The $I_{m,\eta}$-equivariant (shifted perverse) sheaf $\mathcal{A}^+_{\mu,m} := \pi_*\pi^*\mathcal{B}^+_\mu$ on $\mathcal{M}^t_{0,m,G,\eta}$ is the correct analogue of $\mathcal{A}_\mu$ for the $\Gamma_1(p)$-level (and will make Theorem 10.12 hold). Note that our notation differs slightly from that of [HLS], where $\mathcal{A}^+_\mu$ denotes what we write as $\mathcal{B}^+_\mu$. Also, the subscript m is necessary here, since unlike the i w hao ri case, as explained in §8.2, the sheaf $\mathcal{A}^+_{\mu,m}$ genuinely depends on m.

Working directly with $\mathcal{A}^+_{\mu,m}$ is technically difficult, so we pass to its monodromic pieces. We recall some facts about monodromic sheaves in the sense of [HLS, Sec. 14]. Let

$$\chi :\ T(\mathbb{F}_p) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$$

be a general character. Since $(p-1) : T \to T$ over $\mathrm{Spec}(\mathbb{Z}_p)$ is a connected finite étale cover with covering group $T(\mathbb{F}_p)$, the character χ defines a rank-one $\overline{\mathbb{Q}}_\ell$-local system on T. Following [HLS], we denote by $\mathcal{F}_\chi$ the rank-one local system corresponding to χ^{-1} (note the inverse). For each ring $\mathbf{k} \in \{\mathbb{Z}_p, \overline{\mathbb{Z}}_p, \mathbb{Q}_p, \overline{\mathbb{Q}}_p, \mathbb{F}_p, \overline{\mathbb{F}}_p\}$, we write $\mathcal{F}_{\chi_\mathbf{k}} := \mathcal{F}_{\chi,\mathbf{k}}$ for the base change to $T_\mathbf{k}$. We suppress the index $\mathbf{k}$ when clear.

Lemma 8.6. *For any* $\mathbf{k}$ *as above,*

$$(p-1)_*\overline{\mathbb{Q}}_\ell \;=\; \bigoplus_{\chi \in T(\mathbb{F}_p)^\vee} \mathcal{F}_{\chi_\mathbf{k}} \quad \textit{on } T_\mathbf{k}.$$

Proof. This is [HLS, Lemma 14.1.2(i)]. □

Lemma 8.7. *For any* $\mathbf{k}$ *as above, the local system* $\mathcal{F}_{\chi_\mathbf{k}}$ *is multiplicative:*

$$(m_\mathbf{k})^*\mathcal{F}_{\chi_\mathbf{k}} \;=\; \mathcal{F}_{\chi_\mathbf{k}} \boxtimes \mathcal{F}_{\chi_\mathbf{k}},$$

where $m_\mathbf{k} : T_\mathbf{k} \times T_\mathbf{k} \to T_\mathbf{k}$ *is multiplication and* $\boxtimes$ *is the external product.*

Proof. This is [HLS, Lemma 14.1.2(ii)]. □

We recall the definition of χ-monodromic complexes [HLS, Sec. 16.2]. Let X be a finite-type separated $\mathbf{k}$-scheme with a T-action $a : T \times X \to X$.

Definition 8.8. A complex $K \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$ is (*strongly*) χ-*monodromic* if there exists an isomorphism

$$\theta :\ a^*K \xrightarrow{\sim} \mathcal{F}_\chi \boxtimes K,$$

such that:

(i) it is rigidified on the unit section, i.e. $\theta|_{\{1\}\times X} = \mathrm{id}_K$;
(ii) it satisfies the cocycle condition, namely the diagram

$$\begin{tikzcd}
(\mathrm{id}_T \times a)^*(\mathcal{F}_\chi \boxtimes K) \arrow[rr, equal] & & \mathcal{F}_\chi \boxtimes a^*K \arrow[d, "\wr"', "\mathrm{id}\boxtimes\theta"] \\
\big(a \circ (\mathrm{id}_T \times a)\big)^*K \arrow[u, "(\mathrm{id}_T\times a)^*(\theta)", "\wr"'] \arrow[d, equal] & & \mathcal{F}_\chi \boxtimes \mathcal{F}_\chi \boxtimes K \arrow[d, equal, "(*)"] \\
\big(a \circ (m \times \mathrm{id})\big)^*K \arrow[rr, "(m\times\mathrm{id})^*(\theta)", "\sim"'] & & (m \times \mathrm{id})^*(\mathcal{F}_\chi \boxtimes K).
\end{tikzcd}$$

commutes, where $(*)$ uses Lemma 8.7.

Remark 8.9. When χ is trivial, Definition 8.8 reduces to the notion of T-equivariance. If K is shifted perverse and $\mathbf{k}$ is a field, property (i) implies (ii) [HLS, Remark 16.2.2]. In this paper, K will always be shifted perverse.

By Lemma 8.7 and Definition 8.8, the sheaf $\mathcal{F}_\chi$ is itself χ-monodromic on T, so Lemma 8.6 decomposes $(p-1)_*\overline{\mathbb{Q}}_\ell$ into χ-monodromic sheaves as χ varies.

More generally, let $\pi : Y \to X$ be a finite étale morphism of schemes with T-actions, where X is as above and Y has the same properties, and suppose π is *equivariant* in the sense that for $t \in T$ and $y \in Y$,

$$\pi(t \cdot y) = t^{p-1} \cdot \pi(y). \tag{8.3}$$

If moreover π is a connected finite étale cover with covering group $T(\mathbb{F}_p)$, then for any $K \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$ there is a decomposition

$$\pi_*\pi^*K \;=\; \bigoplus_\chi K_\chi,$$

where each K_χ is χ-monodromic; see [HLS, Sec. 16.3]. In particular, for $Y = \mathcal{M}^t_{1,m,G,\eta}$, $X = \mathcal{M}^t_{0,m,G,\eta}$, and $K = \mathcal{B}^+_\mu$, all hypotheses hold, and we obtain a decomposition of $\mathcal{A}^+_{\mu,m} = \pi_*\pi^*(\mathcal{B}^+_\mu)$ into χ-monodromic shifted perverse sheaves:

$$\mathcal{A}^+_{\mu,m} = \pi_*\pi^*(\mathcal{B}^+_\mu) \;=\; \bigoplus_\chi \mathcal{A}^+_{\mu,m,\chi}. \tag{8.4}$$

We can describe $\mathcal{A}^+_{\mu,m,\chi}$ concretely using Lemma 8.3.

Lemma 8.10. *Via the identification $\mathcal{M}^t_{0,m,G,\eta} = \mathcal{M}_{0,m,G,\eta} \times T$ given by $\mathrm{can}_{1,m}$ in Lemma 7.4, there is a natural isomorphism*

$$\mathcal{A}^+_{\mu,m,\chi} \;=\; \mathcal{A}_\mu \boxtimes \mathcal{F}_\chi.$$

Proof. By Lemma 8.3, the morphism $\pi : \mathcal{M}^t_{1,m,G,\eta} \to \mathcal{M}^t_{0,m,G,\eta}$ is $\mathrm{id} \times (p-1) : \mathcal{M}_{0,m,G,\eta} \times T \to \mathcal{M}_{0,m,G,\eta} \times T$. Hence

$$\pi_*\pi^*(\mathcal{B}^+_\mu) = \pi_*\pi^*(\mathcal{A}_\mu \boxtimes \overline{\mathbb{Q}}_\ell) = \mathcal{A}_\mu \boxtimes (p-1)_*\overline{\mathbb{Q}}_\ell \overset{\text{Lemma 8.6}}{=} \mathcal{A}_\mu \boxtimes \Big(\bigoplus_\chi \mathcal{F}_\chi\Big) = \bigoplus_\chi \mathcal{A}_\mu \boxtimes \mathcal{F}_\chi.$$

∎

8.2. **Dependence of $\mathcal{A}^+_{\mu,m,\chi}$ on m.** In this subsection, we discuss how $\mathcal{A}^+_{\mu,m,\chi}$ on $\mathcal{M}^t_{0,m,G,\eta}$ varies with the truncation index m. Choose $m' = (m'^+, m'^-)$ with $m'^+ \geq m^+$ and $m'^- \leq m^-$.

The closed embeddings $\mathcal{M}_{0,m,G,\eta} \subset \mathrm{Gr}_G$ and $\mathcal{M}_{0,m',G,\eta} \subset \mathrm{Gr}_G$ (Lemma 4.4) are compatible: they factor as closed embeddings $\mathcal{M}_{0,m,G,\eta} \subset \mathcal{M}_{0,m',G,\eta} \subset \mathrm{Gr}_G$, where the inclusion $\rho : \mathcal{M}_{0,m,G,\eta} \subset \mathcal{M}_{0,m',G,\eta}$ is defined since any lattice bounded by m is also bounded by m'. Moreover, ρ is defined over $\mathbb{Z}_p$ by the same argument. By Remark 8.5, the $I_{m,\eta}$-orbit of $\mu(t)$ agrees with the $I_{m',\eta}$-orbit, so $\mathcal{A}_{\mu,m}$ and $\mathcal{A}_{\mu,m'}$ are compatible in the sense that $\rho_*(\mathcal{A}_{\mu,m}) = \mathcal{A}_{\mu,m'}$. Thus for the Iwahori level, there is an unambiguous perverse sheaf $\mathcal{A}_\mu$ compatible with different truncations, and the nearby cycles attached to this sheaf always give the same central function in the Iwahori Hecke algebra.

In contrast, $\mathcal{A}^+_{\mu,m,\chi}$ and $\mathcal{A}^+_{\mu,m',\chi}$ need not be compatible; there can be up to $p-1$ different sheaves.

Lemma 8.11. *There exists a natural closed embedding $\rho^t : \mathcal{M}^t_{0,m,G} \subset \mathcal{M}^t_{0,m',G}$ such that*

$$\begin{tikzcd} \mathcal{M}^t_{0,m,G} \arrow[r, "\rho^t"] \arrow[d] & \mathcal{M}^t_{0,m',G} \arrow[d] \\ \mathcal{M}_{0,m,G} \arrow[r, "\rho"] & \mathcal{M}_{0,m',G} \end{tikzcd}$$

is Cartesian.

Proof. We need an isomorphism between the determinant line bundles

$$\mathrm{Det}\Big[\frac{\mathcal{W}_{i-1}}{t^{m^+}\mathbb{V}_{i-1}[t]_{\mathcal{O}_S}} \to \frac{\mathcal{W}_i}{t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S}}\Big] \quad \text{and} \quad \mathrm{Det}\Big[\frac{\mathcal{W}_{i-1}}{t^{m'^+}\mathbb{V}_{i-1}[t]_{\mathcal{O}_S}} \to \frac{\mathcal{W}_i}{t^{m'^+}\mathbb{V}_i[t]_{\mathcal{O}_S}}\Big].$$

The second complex is the direct sum of the first and the complex

$$\Big[\frac{t^{m^+}\mathbb{V}_{i-1}[t]_{\mathcal{O}_S}}{t^{m'^+}\mathbb{V}_{i-1}[t]_{\mathcal{O}_S}} \to \frac{t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S}}{t^{m'^+}\mathbb{V}_i[t]_{\mathcal{O}_S}}\Big],$$

whose determinant line bundle is canonically trivialized using the basis $e_i \in \mathbb{V}$ [HLS, Lemma 7.2.1]. This yields the required isomorphism. ☐

In the proof of Lemma 8.11, we produced an isomorphism

$$\mathrm{Det}\Big[\frac{\mathcal{W}_{i-1}}{t^{m^+}\mathbb{V}_{i-1}[t]_{\mathcal{O}_S}} \to \frac{\mathcal{W}_i}{t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S}}\Big] \;\cong\; \mathrm{Det}\Big[\frac{\mathcal{W}_{i-1}}{t^{m'^+}\mathbb{V}_{i-1}[t]_{\mathcal{O}_S}} \to \frac{\mathcal{W}_i}{t^{m'^+}\mathbb{V}_i[t]_{\mathcal{O}_S}}\Big].$$

Let a_i (resp. a_i') be the distinguished section of the complex on the left (resp. right). The isomorphism sends a_i to $w_p^{m'^+-m^+}\cdot a_i'$, so it does **not** preserve distinguished sections. Since $w_p^{m'^+-m^+}$ has a $(p-1)$-st root in $\mathbb{Q}_p$ if and only if $m'^+ - m^+$ is divisible by $p-1$, the natural embedding $\mathcal{M}^t_{1,m,G} \subset \mathcal{M}^t_{1,m',G}$ exists if and only if $m'^+ - m^+$ is divisible by $p-1$ (see also [HLS, Sec. 9.6]).

We now compare $\mathcal{A}^+_{\mu,m,\chi}$ and $\mathcal{A}^+_{\mu,m',\chi}$. Recall that $\mathcal{M}^t_{0,m,G,\eta}$ and $\mathcal{M}^t_{0,m',G,\eta}$ admit trivializations via the canonical sections $\mathrm{can}_{1,m}$ and $\mathrm{can}_{1,m'}$ (Lemma 7.4).

Proposition 8.12. *Under the identifications $\mathcal{M}^t_{0,m,G,\eta} = \mathcal{M}_{0,m,G,\eta}\times T$ and $\mathcal{M}^t_{0,m',G,\eta} = \mathcal{M}_{0,m',G,\eta}\times T$ given by $\mathrm{can}_{1,m}$ and $\mathrm{can}_{1,m'}$, the embedding ρ^t of Lemma 8.11 is*

$$\mathcal{M}_{0,m,G,\eta}\times T \xrightarrow{f_{m,m'}\times \mathrm{id}_T} \mathcal{M}_{0,m,G,\eta}\times T\times T \xrightarrow{\rho\times m_T} \mathcal{M}_{0,m',G,\eta}\times T,$$

where $f_{m,m'} = (\mathrm{id}_{\mathcal{M}_{0,m,G,\eta}}, w_p^{m'^+-m^+}) : \mathcal{M}_{0,m,G,\eta} \to \mathcal{M}_{0,m,G,\eta}\times T$, and $w_p^{m'^+-m^+} : \mathcal{M}_{0,m,G,\eta} \to \mathrm{Spec}(\mathbb{Q}_p) \to T$ is induced by $(w_p^{m'^+-m^+},\dots,w_p^{m'^+-m^+}) \in T(\mathbb{Q}_p)$. Consequently,

$$(\rho^t)^*(\mathcal{A}^+_{\mu,m',\chi}) \;=\; \mathcal{A}^+_{\mu,m,\chi}\otimes \mathcal{K}_\chi^{\otimes(m'^+-m^+)},$$

where $\mathcal{K}_\chi$ is the rank-one local system on $\mathrm{Spec}(\mathbb{Q}_p)$ obtained by pulling back $\mathcal{F}_\chi$ along the $\mathbb{Q}_p$-point $(w_p,\dots,w_p)\in T(\mathbb{Q}_p)$.

Proof. As noted above, the isomorphism

$$\mathrm{Det}\Big[\frac{\mathcal{W}_{i-1}}{t^{m^+}\mathbb{V}_{i-1}[t]_{\mathcal{O}_S}} \to \frac{\mathcal{W}_i}{t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S}}\Big] \;\cong\; \mathrm{Det}\Big[\frac{\mathcal{W}_{i-1}}{t^{m'^+}\mathbb{V}_{i-1}[t]_{\mathcal{O}_S}} \to \frac{\mathcal{W}_i}{t^{m'^+}\mathbb{V}_i[t]_{\mathcal{O}_S}}\Big]$$

sends a_i to $w_p^{m'^+-m^+}\cdot a_i'$. By Lemma 7.4, $\mathrm{can}_{1,m}$ (resp. $\mathrm{can}_{1,m'}$) sends a_i (resp. a_i') to the identity section of $\mathcal{O}_S$, so the restriction of $\mathrm{can}_{1,m'}$ to $\mathcal{M}_{0,m,G,\eta}$ sends a_i to $w_p^{m'^+-m^+}$.

By Lemma 8.10, $\mathcal{A}^+_{\mu,m',\chi} = \mathcal{A}_\mu\boxtimes\mathcal{F}_\chi$ on $\mathcal{M}_{0,m',G,\eta}\times T$ and $\mathcal{A}^+_{\mu,m,\chi} = \mathcal{A}_\mu\boxtimes\mathcal{F}_\chi$ on $\mathcal{M}_{0,m,G,\eta}\times T$. Hence

$$\begin{aligned}(f\times\mathrm{id}_T)^*(\rho\times m_T)^*(\mathcal{A}^+_{\mu,m',\chi}) &= (f\times\mathrm{id}_T)^*(\rho\times m_T)^*(\mathcal{A}_\mu\boxtimes\mathcal{F}_\chi)\\ &= (f\times\mathrm{id}_T)^*(\mathcal{A}_\mu\boxtimes\mathcal{F}_\chi\boxtimes\mathcal{F}_\chi)\\ &= (\mathcal{A}_\mu\boxtimes\mathcal{F}_\chi)\otimes\mathcal{K}_\chi^{\otimes(m'^+-m^+)}\\ &= \mathcal{A}^+_{\mu,m,\chi}\otimes\mathcal{K}_\chi^{\otimes(m'^+-m^+)},\end{aligned}$$

where in the second line we used Lemma 8.7. ☐

Remark 8.13. By definition, the rank-one local system $\mathcal{K}_\chi$ in Proposition 8.12 is related to the tamely ramified extension $\mathbb{Q}_p\big(w_p^{1/(p-1)}\big)/\mathbb{Q}_p$. More precisely, $\mathcal{K}_\chi$ corresponds to the character

$$\beta:\ \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow \mathrm{Gal}\big(\mathbb{Q}_p\big(w_p^{\frac{1}{p-1}}\big)/\mathbb{Q}_p\big) \cong \mathbb{F}_p^\times \xrightarrow{\Delta} T(\mathbb{F}_p) \xrightarrow{\chi^{-1}} \overline{\mathbb{Q}}_\ell^\times,$$

where $\mathrm{Gal}\big(\mathbb{Q}_p\big(w_p^{\frac{1}{p-1}}\big)/\mathbb{Q}_p\big)\cong\mathbb{F}_p^\times$ is given by $\sigma\mapsto\big(\sigma(w_p^{\frac{1}{p-1}})/w_p^{\frac{1}{p-1}} \bmod p\mathbb{Z}_p\big)$, and Δ is the diagonal map $x\mapsto(x,\dots,x)$. Note that $\mathcal{K}_\chi^{\otimes(p-1)}\cong\overline{\mathbb{Q}}_\ell$, since $\mathcal{F}_\chi^{\otimes(p-1)}\cong\overline{\mathbb{Q}}_\ell$. This implies that when $m'^+\equiv m^+ \pmod{p-1}$, $(\rho^t)^*(\mathcal{A}^+_{\mu,m',\chi}) = \mathcal{A}^+_{\mu,m,\chi}$.

9. Central elements in Hecke algebras

Using the sheaf $\mathcal{A}^+_{\mu,m,\chi}$, we construct a function $\tau^{ss}_{\mu,m,\chi}$ in the Hecke algebra for the $\Gamma_1(p)$-level. Before doing so, we review some facts about $\Gamma_1(p)$-Hecke algebras

9.1. **Definitions of Hecke algebras and their centers.** For now, we work over the local field $\mathbb{F}_p(\!(t)\!)$, though the results in this subsection extend verbatim to any nonarchimedean local field (see [HLS, Sec. 13] for more details). Recall that in §2 we fixed an Iwahori subgroup of G for every local field (in particular for $\mathbb{F}_p(\!(t)\!)$); we denote it by I. Recall also the pro-p Iwahori subgroup $I^+ \subset I$ which is the pro-unipotent radical of I. The Hecke algebra $\mathcal{H}(G)$ consists of compactly supported, locally constant functions (with values in $\overline{\mathbb{Q}_\ell}$) on $G(\mathbb{F}_p(\!(t)\!))$ (which we denote simply by G later).

The Iwahori Hecke algebra $\mathcal{H}(G, I) \subset \mathcal{H}(G)$ is

$$\mathcal{H}(G, I) := \{\, f \in \mathcal{H}(G) \mid f(i_1 g i_2) = f(g) \ \ \forall\, i_1, i_2 \in I,\ \forall\, g \in G \,\}.$$

Similarly, the pro-p Iwahori (or $\Gamma_1(p)$) Hecke algebra $\mathcal{H}(G, I^+) \subset \mathcal{H}(G)$ is

$$\mathcal{H}(G, I^+) := \{\, f \in \mathcal{H}(G) \mid f(i_1^+ g i_2^+) = f(g) \ \ \forall\, i_1^+, i_2^+ \in I^+,\ \forall\, g \in G \,\}.$$

Via reduction, $I/I^+ \cong B(\mathbb{F}_p)/U(\mathbb{F}_p) \cong T(\mathbb{F}_p)$, so χ gives a character of I, still denoted χ. The following variant of the Hecke algebra can be viewed as the χ-component of $\mathcal{H}(G, I^+)$ (though, strictly speaking, one does not obtain a direct sum decomposition). Define

$$\mathcal{H}(G, I, \chi) := \{\, f \in \mathcal{H}(G) \mid f(i_1 g i_2) = \chi^{-1}(i_1)\, f(g)\, \chi^{-1}(i_2) \ \ \forall\, i_1, i_2 \in I,\ \forall\, g \in G \,\}.$$

Recall we used χ^{-1} in the definition of $\mathcal{F}_\chi$, and we follow the same convention here. Since $I^+ \subset \ker(\chi)$, we have $\mathcal{H}(G, I, \chi) \subset \mathcal{H}(G, I^+)$. When χ is trivial, clearly $\mathcal{H}(G, I, \chi) = \mathcal{H}(G, I)$.

Remark 9.1. There is a natural inclusion $\bigoplus_\chi \mathcal{H}(G, I, \chi) \subset \mathcal{H}(G, I^+)$, but in general equality fails because of the bi-χ^{-1}-equivariance requirement in the definition of $\mathcal{H}(G, I, \chi)$ (i.e. χ^{-1}-equivariance on both sides).

Normalize the Haar measure dx_I on G so that $\mathrm{vol}(I) = 1$. With respect to this measure, convolution $*$ endows all the Hecke algebras above with a multiplication. Denote their centers by $\mathcal{Z}(G)$, $\mathcal{Z}(G, I)$, $\mathcal{Z}(G, I^+)$, and $\mathcal{Z}(G, I, \chi)$. The identity of $\mathcal{H}(G, I, \chi)$ is the idempotent $e_\chi \in \mathcal{H}(G)$: it is supported on I and satisfies $e_\chi(y) = \chi(y)^{-1}$ for $y \in I$. When χ is trivial, e_χ is the characteristic function of I.

A key result for the $\Gamma_1(p)$-Hecke algebra ([Hai12, Lemma 10.0.1]) motivates the use of χ-monodromic sheaves.

Lemma 9.2. *There is a canonical injective homomorphism*

$$\begin{aligned} \mathcal{Z}(G, I^+) &\hookrightarrow \prod_\chi \mathcal{Z}(G, I, \chi) \\ z &\longmapsto (z * e_\chi)_\chi, \end{aligned} \tag{9.1}$$

identifying $\mathcal{Z}(G, I^+)$ with the set of tuples $(z^\chi)_\chi$ such that for any χ, any $w \in W$, and any extension $\widetilde{\chi} : T(\mathbb{F}_p(\!(t)\!)) \to \overline{\mathbb{Q}_\ell}^\times$ of χ, the scalar by which $z^\chi\, dx_I$ acts on $i_B^G(\widetilde{\chi})^\chi$ coincides with the scalar by which $z^{{}^w\chi}\, dx_I$ acts on $i_B^G(\widetilde{\chi})^{{}^w\chi}$. Here, as above, we view χ as a character of I.

Proof. This is [Hai12, Lemma 10.0.1]. □

In particular, any $z \in \mathcal{Z}(G, I^+) \subset \mathcal{H}(G, I^+)$ can be written uniquely as a sum $z = \sum_\chi z_\chi$ with $z_\chi := z * e_\chi \in \mathcal{Z}(G, I, \chi) \subset \mathcal{H}(G, I^+)$.

9.2. **Construction of central functions via nearby cycles.** We recall some facts about nearby cycles following [HR21, Sec. 6.0.1].

For a separated $\mathbb{F}_p$-scheme X of finite type, let $D_c^b(X \times_s \eta)$ be the bounded derived category of $\overline{\mathbb{Q}}_\ell$-complexes on $X_{\overline{\mathbb{F}}_p}$ with constructible cohomology, equipped with a continuous action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ compatible with its action on $X_{\overline{\mathbb{F}}_p}$. If $f : X \hookrightarrow Y$ is a closed immersion of $\mathbb{F}_p$-schemes inducing an isomorphism on reduced loci, then by topological invariance of étale cohomology [StaPro, Tag 03SI], f induces an equivalence $D_c^b(X \times_s \eta) \cong D_c^b(Y \times_s \eta)$.

For a separated $\mathbb{Z}_p$-scheme X of finite type, the nearby cycles functor

$$R\Psi_X : D_c^b(X_\eta) \longrightarrow D_c^b(X \times_s \eta)$$

is defined by $R\Psi_X(\mathcal{A}) = \bar{i}^*\bar{j}_*\mathcal{A}_{\bar{\eta}}$, where $\bar{j} : X_{\bar{\eta}} \hookrightarrow X_{\overline{\mathbb{Z}}_p}$ and $\bar{i} : X_{\bar{s}} \hookrightarrow X_{\overline{\mathbb{Z}}_p}$ are the embeddings of geometric fibers. If $f : X \to Y$ is a morphism of $\mathbb{Z}_p$-schemes of finite type, there are natural transformations

$$f_{\bar{s},!} \circ R\Psi_X \longrightarrow R\Psi_Y \circ f_{\eta,!} \qquad \big(\text{resp. } f_{\bar{s}}^* \circ R\Psi_Y \longrightarrow R\Psi_X \circ f_\eta^*\big),$$

which are isomorphisms when f is proper (resp. smooth). Nearby cycles commute with external products:

$$R\Psi_{X\times_{\mathbb{Z}_p}Y} \cong R\Psi_X \boxtimes R\Psi_Y.$$

See [SGA7, Exp. XIII] and [Il] for details.

Define

$$\tau^{ss}_{\mu,m,\chi} : \mathcal{M}^t_{0,m,G}(\mathbb{F}_p) \longrightarrow \overline{\mathbb{Q}}_\ell, \qquad x \longmapsto (-1)^{d_\mu}\,\mathrm{Tr}^{ss}\Big(\Phi \,\Big|\, \big(R\Psi_{\mathcal{M}^t_{0,m,G}}(\mathcal{A}^+_{\mu,m,\chi})\big)_{\bar{x}}\Big),$$

where Tr^{ss} denotes the semisimple trace in the sense of Rapoport (see [HN, Sec. 3.1] and [PZ, Sec. 10.4]), $d_\mu = \dim(O_\mu)$ is the dimension of the $I_{m,\eta}$-orbit, and $\Phi \in \mathcal{W}_{\mathbb{Q}_p}$ is any geometric Frobenius element. Later we show that $\mathcal{A}^+_{\mu,m,\chi}$ is $I_{m,\eta}$-equivariant (Lemma 10.1). Via the embedding $\mathcal{M}^t_{0,m,G,s} \subset \mathrm{Fl}^t_G$ (Lemma 6.2), we regard $\tau^{ss}_{\mu,m,\chi}$ as a function on $\mathrm{Fl}^t_{G,\mathbb{F}_p}(\mathbb{F}_p) = G/I^+$, and we keep the same notation. The $I_{m,\eta}$-equivariance together with χ-monodromy implies $\tau^{ss}_{\mu,m,\chi} \in \mathcal{H}(G, I, \chi)$ ([HLS, Proposition 16.6.2]).

Remark 9.3. The factor $(-1)^{d_\mu}$, together with normalizations used later, makes the equality in Theorem 10.10 hold exactly. Without these normalizations, one only obtains equality up to a scalar depending on μ.

Similarly, set

$$\tau^{ss}_{\mu,m} : \mathcal{M}^t_{0,m,G}(\mathbb{F}_p) \to \overline{\mathbb{Q}}_\ell, \quad x \longmapsto (-1)^{d_\mu}\,\mathrm{Tr}^{ss}\Big(\Phi \mid (R\Psi_{\mathcal{M}^t_{0,m,G}}(\mathcal{A}^+_{\mu,m})_{\bar{x}}\Big),$$

which lies in $\mathcal{H}(G, I^+)$. Using the decomposition

$$\mathcal{A}^+_{\mu,m} = \bigoplus_{\chi\in T(\mathbb{F}_p)^\vee} \mathcal{A}^+_{\mu,m,\chi} \quad (8.4),$$

we obtain

$$\tau^{ss}_{\mu,m} = \sum_\chi \tau^{ss}_{\mu,m,\chi} \in \mathcal{H}(G, I^+).$$

We expect $\tau^{ss}_{\mu,m} \in \mathcal{Z}(G, I^+)$ (shown in Theorem 10.12). By Lemma 9.2, each $\tau^{ss}_{\mu,m,\chi}$ should lie in $\mathcal{Z}(G, I, \chi)$—this is proved in [HLS, Sec. 16]. Note, however, that centrality of each $\tau^{ss}_{\mu,m,\chi}$ does not by itself imply centrality of the sum $\tau^{ss}_{\mu,m}$; one must also verify the Weyl group compatibility stated in Lemma 9.2. We will check these compatibilities by identifying $\tau^{ss}_{\mu,m,\chi}$ with (the translation by central elements of) certain functions $z^{ss}_{\mu,\chi}$ for which the compatibility is built into the definition.

9.3. **Construction of central functions via Bernstein varieties.** In this section, we construct central elements in $\mathcal{H}(G, I^+)$ and $\mathcal{H}(G, I, \chi)$ (see [HLS, Sec. 13] for details). Let $\Phi_t \in \mathcal{W}_{\mathbb{F}_p(\!(t)\!)}$ be a geometric Frobenius corresponding to $t \in \mathbb{F}_p(\!(t)\!)$ via the Artin map. We work over $\mathbb{F}_p(\!(t)\!)$, but the construction adapts to any local field F.

Recall that the Bernstein center $\mathfrak{Z}(G)$ is the algebra of endomorphisms of the identity functor on smooth $G(\mathbb{F}_p(\!(t)\!))$-representations; equivalently, it is the algebra of regular functions on the Bernstein variety (whose points are equivalence classes of irreducible smooth $G(\mathbb{F}_p(\!(t)\!))$-representations). We specify $Z \in \mathfrak{Z}(G)$ as a regular function on this variety: for irreducible π, we write $Z(\pi) \in \overline{\mathbb{Q}}_\ell$. Elements of $\mathfrak{Z}(G)$ are also viewed as G-invariant, essentially compact distributions on $\mathcal{H}(G)$. Using the measure dx_I with $\mathrm{vol}(I) = 1$, any $f \in \mathcal{H}(G)$ defines an essentially compact distribution $f\,dx_I$, and convolution is compatible: $(f * g)\,dx_I = f\,dx_I * g\,dx_I$. For $Z \in \mathfrak{Z}(G)$, let $1_I \in \mathcal{Z}(G, I)$ (resp. $1_{I^+} \in \mathcal{Z}(G, I^+)$) be the characteristic function of I (resp. I^+). Then there exists a unique $g \in \mathcal{Z}(G, I)$ (resp. $g \in \mathcal{Z}(G, I^+)$) with $Z * 1_I\,dx_I = g\,dx_I$ (resp. $Z * 1_{I^+}\,dx_I = g\,dx_I$). See [Hai14, Sec. 3] for more details on these equivalent descriptions.

We first give a construction of central elements using semisimple local Langlands parameters. Let (r, V) be an algebraic representation of ${}^LG := \widehat{G} \rtimes \mathcal{W}_{\mathbb{F}_p(\!(t)\!)}$ (with $\widehat{G}$ over $\overline{\mathbb{Q}}_\ell$ being the dual group of G). There is an associated distribution $Z_V \in \mathfrak{Z}(G)$. For an irreducible smooth π, let $\varphi_\pi : \mathcal{W}_{\mathbb{F}_p(\!(t)\!)} \to {}^L G$ be the semisimple Langlands parameter of [FS]. Define

$$Z_V(\pi) = \mathrm{Tr}\big(r\varphi_\pi(\Phi_t) \mid V^{\,r\varphi_\pi(\mathcal{I}_{\mathbb{F}_p(\!(t)\!)})}\big), \tag{9.2}$$

where $\mathcal{I}_{\mathbb{F}_p(\!(t)\!)}$ is the inertia group of $\mathbb{F}_p(\!(t)\!)$. Since we take the semisimple trace (i.e. on $\mathcal{I}_{\mathbb{F}_p(\!(t)\!)}$-invariants), $Z_V(\pi)$ does not depend on the choice of Φ_t. This yields a distribution in $\mathfrak{Z}(G)$ by [Hai14, Proposition 5.7.1, Remark 5.7.2]. For the dominant $\mu \in X_*(T)^+$ fixed earlier in §8.1, let V_μ be the corresponding highest-weight representation of ${}^L G$ (when G splits, W acts trivially on V_μ and V_μ is the usual highest-weight $\widehat{G}$-module; see [Hai14, Sec. 6.1]). Set

$$\begin{aligned} z_\mu^{ss} &:= Z_{V_\mu} * 1_{I^+}\, dx_I \in \mathcal{Z}(G, I^+), \\ z_{\mu,\chi}^{ss} &:= Z_{V_\mu} * e_\chi\, dx_I \in \mathcal{Z}(G, I, \chi). \end{aligned}$$

From

$$1_{I^+} = \frac{1}{|T(\mathbb{F}_p)|} \sum_\chi e_\chi,$$

we obtain

$$z_\mu^{ss} = \frac{1}{|T(\mathbb{F}_p)|} \sum_\chi z_{\mu,\chi}^{ss}. \tag{9.3}$$

Remark 9.4. In [HR21, Main Theorem], an analogous construction in $\mathcal{H}(G, I)$ uses the Satake parameter $s(\pi)$ (for Iwahori-spherical π) in place of $\varphi_\pi(\Phi_t)$. Since the support of $Z_{V_\mu} * 1_I dx_I$ lies in the Iwahori-spherical block, it suffices to specify a scalar on each such representation. The construction agrees with $Z_{V_\mu} * 1_I\, dx_I$, as [Li] shows $\varphi_\pi(\Phi_t) = s(\pi)$ and $\varphi_\pi(\mathcal{I}_{\mathbb{F}_p(\!(t)\!)}) = 1 \rtimes \mathcal{I}_{\mathbb{F}_p(\!(t)\!)}$ up to $\widehat{G}$-conjugacy.

We also give a construction via Bushnell–Kutzko types ([BK]). Since $z_\mu^{ss}, z_{\mu,\chi}^{ss} \in \mathcal{H}(G, I^+)$, their support sits in the depth-zero block (representations with nonzero I^+-fixed vectors). Thus they are determined by their values on equivalence classes of irreducible depth-zero representations.

Let $\widetilde{\chi} : T(\mathbb{F}_p(\!(t)\!)) \to \overline{\mathbb{Q}}_\ell^\times$ be given by $\widetilde{\chi}(a\,\nu(t)) = \chi(\bar{a})$, for $a \in T(\mathbb{F}_p[\![t]\!])$, $\nu \in X_*(T)$, and $\bar{a}$ the reduction of a. Since (I, χ) is a Bushnell–Kutzko type for the cuspidal pair $(T(\mathbb{F}_p(\!(t)\!)), \widetilde{\chi})$ ([Hai12, Proposition 3.3.1]), any $z \in \mathcal{Z}(G, I, \chi)$ is uniquely determined by the scalar by which it acts on $i_B^G(\widetilde{\chi}\eta)^\chi$ as η ranges over unramified characters of $T(\mathbb{F}_p(\!(t)\!))$. (This scalar is the value of z on any irreducible subquotient of $i_B^G(\widetilde{\chi}\eta)$.)

By definition, $z_{\mu,\chi}^{ss}$ is the unique element of $\mathcal{Z}(G, I, \chi)$ such that $z_{\mu,\chi}^{ss}\, dx_I$ acts on $i_B^G(\widetilde{\chi}\eta)^\chi$ by

$$\mathrm{Tr}\Big(r\varphi_{\widetilde{\chi}\eta}(\Phi_t) \mid V_\mu^{\,r\varphi_{\widetilde{\chi}\eta}(\mathcal{I}_{\mathbb{F}_p(\!(t)\!)})}\Big),$$

for any unramified η. Here $\varphi_{\widetilde{\chi}\eta}$ is the parameter attached to $\widetilde{\chi}\eta$ via local class field theory, compatible with [FS] by the compatibility with parabolic induction ([FS, Theorem I.9.6]).

Similarly, $z_\mu^{ss} \in \mathcal{Z}(G, I^+)$ is characterized by: $z_\mu^{ss}\, dx_{I^+}$ acts on $i_B^G(\widetilde{\chi}\eta)^\chi$ by the same scalar, for all χ, η (here dx_{I^+} has $\mathrm{vol}(I^+) = 1$).

Conversely, these characterizations can serve as definitions of our central functions. Only local class field theory is needed to define $\varphi_{\widetilde{\chi}\eta}$. One checks directly that the $z_{\mu,\chi}^{ss}$ are regular and satisfy the compatibility in Lemma 9.2 ([HLS, Sec. 13.4]); taking average over χ then yields $z_\mu^{ss} \in \mathcal{Z}(G, I^+)$.

When μ is minuscule, one can give explicit values of z_μ^{ss} and $z_{\mu,\chi}^{ss}$ on I^+–double cosets ([HLS, Propositions 13.5.2]). For our later use, we record the special case $G = T$ (where every cocharacter is minuscule).

Lemma 9.5. *Let $G = T$. Then $z_{\mu,\chi}^{ss} \in \mathcal{Z}(G, I, \chi)$ is the unique function supported on $I\,\mu(t)\,I$ such that for $a \in T(\mathbb{F}_p[\![t]\!])$,*

$$z_{\mu,\chi}^{ss}\big(I^+\, a\, \mu(t)\, I^+\big) = \begin{cases} \chi(\bar{a})^{-1}, & \text{if } \chi \circ \mu(\mathbb{F}_p^\times) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. This is the case $G = T$ of [HLS, Proposition 13.5.2(i)]. One can also check directly that this function acts on depth-zero characters by the required scalars. ∎

10. The equation between central elements

We now work toward one of our final goals: to show that the functions $\tau^{ss}_{\mu,m,\chi}$ and $z^{ss}_{\mu,\chi}$, though constructed in different ways, are actually equal (up to a translation by some specific central element as we will explain later) in $\mathcal{Z}(G, I, \chi)$. This gives an explicit description of $\tau^{ss}_{\mu,m,\chi}$ as a regular function on the Bernstein variety.

10.1. **Constant term homomorphism and geometric constant terms.** For general μ, it is difficult to verify $\tau^{ss}_{\mu,m,\chi} = z^{ss}_{\mu,\chi}$ in $\mathcal{Z}(G, I, \chi)$ directly, since computing their values on I^+-double cosets is hard. Instead, following the strategy of the proof of the test function conjecture for parahoric local models in [HR21], we compare them after applying the constant term homomorphism

$$c_T^G : \ \mathcal{Z}(G, I, \chi) \longrightarrow \mathcal{Z}(T, I_T, \chi),$$

where I_T is the unique Iwahori subgroup $T(\mathbb{F}_p[\![t]\!])$ of $T(\mathbb{F}_p(\!(t)\!))$. The constant term homomorphism is induced by a surjective morphism from the depth-zero block of T to that of G ([Hai12, Sec. 5.1]) and is therefore injective on the level of coordinate rings. Since c_T^G is injective, it suffices to check that

$$c_T^G(\tau^{ss}_{\mu,m,\chi}) = c_T^G(z^{ss}_{\mu,\chi}).$$

There is a more concrete description of c_T^G via integrals ([Hai12, Sec. 5.4]):

$$c_T^G(f)(m) \ = \ \delta_B^{1/2}(m) \cdot \int_{U(\mathbb{F}_p(\!(t)\!))} f(mu)\, du,$$

where δ_B is the modulus character of B and du is normalized so that $U(\mathbb{F}_p[\![t]\!])$ has volume 1. It is convenient to use a normalization that matches the geometric picture. Consider the Kottwitz homomorphism for T,

$$\kappa_T : T(\mathbb{F}_p(\!(t)\!)) \to T(\mathbb{F}_p(\!(t)\!))/I_T \cong X_*(T), \quad m \mapsto \nu_m,$$

and let $2\rho_B$ be the sum of positive roots determined by B. The *normalized constant term* is

$${}^p c_T^G(f)(m) \ := \ (-1)^{\langle 2\rho_B, \nu_m \rangle}\, c_T^G(f)(m),$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing $X^*(T) \times X_*(T) \to \mathbb{Z}$.

There is a geometric realization of ${}^p c_T^G$ using hyperbolic localization. Recall the diagram 5.2 attached to a $\mathbb{G}_m$-equivariant morphism X/S (with $\mathbb{G}_m$-action étale locally linearizable):

$$\begin{tikzcd} & X^{\pm} \arrow[dl, "q^{\pm}"'] \arrow[dr, "p^{\pm}"] & \\ X^0 & & X. \end{tikzcd}$$

Using this diagram, define functors (pull–push) as in [Ri19, Definition 2.1]:

$$L^+_{X/S} \ := \ (q^+)_! \circ (p^+)^*, \qquad L^-_{X/S} \ := \ (q^-)_* \circ (p^-)^!.$$

First take $X/S = \mathcal{M}_{0,m,G,\eta}/\mathrm{Spec}(\mathbb{Q}_p)$ (resp. $X/S = \mathcal{M}^t_{0,m,G,\eta}/\mathrm{Spec}(\mathbb{Q}_p)$) and apply these functors to $\mathcal{A}_\mu$ (resp. $\mathcal{A}^+_{\mu,m,\chi}$). By [Ri19, Theorem 2.6], there is a natural transformation $L^-_{X/S} \to L^+_{X/S}$ which is an isomorphism on $\mathbb{G}_m$-equivariant bounded-below complexes. The next lemma shows this applies in our situation.

Lemma 10.1. *Both $\mathcal{A}_\mu$ and $\mathcal{A}^+_{\mu,m,\chi}$ are $I_{m,\eta}$-equivariant and hence $\mathbb{G}_{m,\eta}$-equivariant. Their nearby cycles are $I_{m,\bar{s}}$-equivariant and hence $\mathbb{G}_{m,\bar{s}}$-equivariant.*

Proof. We have already seen that $\mathcal{A}_\mu$ is $I_{m,\eta}$-equivariant (see Remark 8.5 and the preceding discussion). For $\mathcal{A}^+_{\mu,m,\chi}$, use the trivialization in Lemma 7.4: by Lemma 8.10, $\mathcal{A}^+_{\mu,m,\chi} = \mathcal{A}_\mu \boxtimes \mathcal{F}_\chi$ and I_m acts only on the first factor, so $\mathcal{A}^+_{\mu,m,\chi}$ is $I_{m,\eta}$-equivariant. Another proof is [HLS, Lemma 16.5.3], which works more generally. For nearby cycles, equivariance follows because I_m is smooth over $\mathrm{Spec}(\mathbb{Z}_p)$ ([HLS, Proposition 6.4.6]). □

When working with $\mathcal{A}_\mu$, $\mathcal{A}^+_{\mu,m,\chi}$, and their nearby cycles, we may focus on $L^+_{X/S}$. We normalize $L^+_{X/S}$ to align with the geometric Satake correspondence. By Proposition 5.11, the "+"-diagram in our case is

$$\begin{tikzcd} & \mathcal{M}_{0,m,B,\eta} \arrow[dl, "q^+"'] \arrow[dr, "p^+"] & \\ \mathcal{M}_{0,m,T,\eta} & & \mathcal{M}_{0,m,G,\eta}. \end{tikzcd}$$

The connected components of $\mathcal{M}_{0,m,T,\eta}$ are indexed by $\nu \in X_*(T) \cong \mathbb{Z}^n$ with $m^- \le \nu_i \le m^+$ (denoted $\nu \le m$), and each component is a single (possibly infinitesimal) point (using $k(\!(t)\!)^\times/k[\![t]\!]^\times \cong \mathbb{Z}$). Write $\mathcal{M}_{0,m,T,\eta,\nu}$ for the component indexed by ν, and let $\mathcal{M}_{0,m,B,\eta,\nu} := (q^+)^{-1}(\mathcal{M}_{0,m,T,\eta,\nu})$.

For $\mathcal{A} \in D^b_c(\mathcal{M}_{0,m,G,\eta})$, compute on each component:

$$L^+_{X/S}(\mathcal{A})\big|_{\mathcal{M}_{0,m,T,\eta,\nu}} \;=\; (q^+_\nu)_!\,(p^+_\nu)^*(\mathcal{A}) \;=:\; L^+_{X/S,\nu}(\mathcal{A}),$$

where q^+_ν, p^+_ν are the restrictions

$$\begin{tikzcd} & \mathcal{M}_{0,m,B,\eta,\nu} \arrow[dl, "q^+_\nu"'] \arrow[dr, "p^+_\nu"] & \\ \mathcal{M}_{0,m,T,\eta,\nu} & & \mathcal{M}_{0,m,G,\eta}. \end{tikzcd}$$

Define the normalized functor

$$\mathrm{CT}_\eta \;:=\; \bigoplus_{\nu \le m} L^+_{X/S,\nu}\langle\langle 2\rho_B, \nu\rangle\rangle,$$

where the outer $\langle\cdot\rangle$ is the twist on sheaves (8.2), and the inner $\langle\cdot,\cdot\rangle$ is the pairing $X^*(T)\times X_*(T) \to \mathbb{Z}$. This equals the normalization in [HR21, Definition 3.15] (there ν is unbounded and components are grouped by $\langle 2\rho_B, \nu\rangle$). The functor CT_η is the properly normalized one in the following sense, where Sat_G (resp. Sat_T) is the Satake category of G (resp. T), and ω_G (resp. ω_T) is a Tannakian fiber functor [HR21, Definition 3.8].

Lemma 10.2. *For any $\mathcal{A} \in \mathrm{Sat}_G$, we have $\mathrm{CT}_\eta(\mathcal{A}) \in \mathrm{Sat}_T$. Moreover, there is a commutative diagram of neutral Tannakian categories*

$$\begin{tikzcd} \mathrm{Sat}_G \arrow[r, "\mathrm{CT}_\eta"] \arrow[d, "\omega_G"'] & \mathrm{Sat}_T \arrow[d, "\omega_T"] \\ \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L G) \arrow[r, "\mathrm{res}"] & \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L T), \end{tikzcd}$$

where res *is the restriction of representations.*

Proof. This is [HR21, Theorem 3.16]. □

Over the geometric special point $\mathrm{Spec}(\overline{\mathbb{F}}_p)$, by Proposition 5.13 and since we only need the part of the complex on the open and closed subcheme $\mathcal{M}_{0,m,T,\bar{s}}$, we may work with

$$\begin{tikzcd} & \mathcal{M}_{0,m,B,\bar{s}} \arrow[dl, "q^+"'] \arrow[dr, "p^+"] & \\ \mathcal{M}_{0,m,T,\bar{s}} & & \mathcal{M}_{0,m,G,\bar{s}}. \end{tikzcd}$$

Again the connected components of $\mathcal{M}_{0,m,T,\bar{s}}$ are indexed by $\nu \le m$, so we define $\mathrm{CT}_{\bar{s}}$ by normalizing $L^+_{X/S}$ in exactly the same way. Since $\mathcal{M}^t_{0,m,T}$ is a T-torsor over $\mathcal{M}_{0,m,T}$, the connected components of $\mathcal{M}_{0,m,T,\eta}$ and $\mathcal{M}_{0,m,T,\bar{s}}$ are indexed by the same set. By Propositions 7.5 and 7.7, we likewise define CT^t_η and $\mathrm{CT}^t_{\bar{s}}$.

Following [HR21], we call $\mathrm{CT}_{\bar{s}}$ and $\mathrm{CT}^t_{\bar{s}}$ the *geometric constant terms*, as justified by the next lemma.

Lemma 10.3. *Via the embeddings $\mathcal{M}_{0,m,G,\bar{s}} \subset \mathrm{Fl}_G$ (Lemma 4.6) and $\mathcal{M}_{0,m,T,\bar{s}} \subset \mathrm{Fl}_T$ (Lemma 4.10), the functor $\mathrm{CT}_{\bar{s}}$ realizes ${}^p c_T^G$ on functions: for any $\mathcal{A} \in D_c^b(\mathcal{M}_{0,m,G,\eta} \times_s \eta)$ whose semisimple trace function*

$$\tau_{\mathcal{A}}^{ss} : \ \mathcal{M}_{0,m,G}(\mathbb{F}_p) \to \overline{\mathbb{Q}}_\ell, \quad x \mapsto \mathrm{Tr}^{ss}(\Phi \mid (\mathcal{A}_{\bar{x}}))$$

lies in $\mathcal{Z}(G, I)$, we have

$${}^p c_T^G(\tau_{\mathcal{A}}^{ss}) \ = \ \tau_{\mathrm{CT}_{\bar{s}}(\mathcal{A})}^{ss},$$

*where $\tau_{\mathrm{CT}_{\bar{s}}(\mathcal{A})}^{ss}$ is the semisimple trace function for $\mathrm{CT}_{\bar{s}}(\mathcal{A}) \in D_c^b(\mathcal{M}_{0,m,T,\eta} \times_s \eta)$. (There is **no** $(-1)^{d_\mu}$ factor.)*

The same holds for $\mathrm{CT}_{\bar{s}}^t$ using the embeddings $\mathcal{M}_{0,m,G,\bar{s}}^t \subset \mathrm{Fl}_G^t$ and $\mathcal{M}_{0,m,T,\bar{s}}^t \subset \mathrm{Fl}_T^t$ (Lemma 6.2).

Proof. For $\mathrm{CT}_{\bar{s}}$, interpret $(p^+)^*$ as restriction and $(q^+)_!$ as integration of functions; then check that the normalization in ${}^p c_T^G$ matches that of $\mathrm{CT}_{\bar{s}}$, as in [HR21, Lemma 7.2]. The proof for $\mathrm{CT}_{\bar{s}}^t$ is identical. □

10.2. **Proof of the main theorem.** We will show that the semisimple trace function attached to $\mathrm{CT}_{\bar{s}}^t\big(R\Psi_{\mathcal{M}_{0,m,G}^t}(\mathcal{A}_{\mu,m,\chi}^+)\big)$ is precisely $c_T^G(\tau_{\mu,m,\chi}^{ss})$ (Lemma 10.9). Suppose for now this is known. To compute $c_T^G(\tau_{\mu,m,\chi}^{ss})$, we need an explicit description of $\mathrm{CT}_{\bar{s}}^t\big(R\Psi_{\mathcal{M}_{0,m,G}^t}(\mathcal{A}_{\mu,\chi}^+)\big)$. The key input is that geometric constant terms commute with nearby cycles on bounded below $\mathbb{G}_m$-equivariant complexes.

Lemma 10.4. *There is a natural isomorphism*

$$\mathrm{CT}_{\bar{s}}^t\big(R\Psi_{\mathcal{M}_{0,m,G}^t}(\mathcal{A}_{\mu,m,\chi}^+)\big) \ = \ R\Psi_{\mathcal{M}_{0,m,T}^t}\big(\mathrm{CT}_\eta^t(\mathcal{A}_{\mu,m,\chi}^+)\big).$$

Proof. By Lemma 10.1, this is the special case of [Ri19, Th. 3.3] asserting that nearby cycles commute with geometric constant terms for bounded below $\mathbb{G}_m$-equivariant complexes. □

We now describe $R\Psi_{\mathcal{M}_{0,m,T}^t}\big(\mathrm{CT}_\eta^t(\mathcal{A}_{\mu,m,\chi}^+)\big)$. The sheaf $\mathrm{CT}_\eta^t(\mathcal{A}_{\mu,m,\chi}^+)$ on $\mathcal{M}_{0,m,T,\eta}^t$ can be characterized using the canonical section $\mathrm{can}_{1,m}$ from Lemma 7.4.

Lemma 10.5. *Via the identification $\mathcal{M}_{0,m,T,\eta}^t = \mathcal{M}_{0,m,T,\eta} \times T$, we have a natural isomorphism*

$$\mathrm{CT}_\eta^t(\mathcal{A}_{\mu,m,\chi}^+) \ = \ \mathrm{CT}_\eta(\mathcal{A}_\mu) \boxtimes \mathcal{F}_\chi.$$

Proof. By Lemma 8.2 we have a commutative diagram with Cartesian squares

$$\begin{tikzcd}
\mathcal{M}_{0,m,T,\eta}^t \arrow[d, "f"] & \mathcal{M}_{0,m,B,\eta}^t \arrow[l, "q^{t,+}"'] \arrow[r, "p^{t,+}"] \arrow[d, "f"] & \mathcal{M}_{0,m,G,\eta}^t \arrow[d, "f"] \\
\mathcal{M}_{0,m,T,\eta} & \mathcal{M}_{0,m,B,\eta} \arrow[l, "q^+"'] \arrow[r, "p^+"] & \mathcal{M}_{0,m,G,\eta}
\end{tikzcd}$$

which, after the identifications in Lemma 7.4, becomes

$$\begin{tikzcd}
\mathcal{M}_{0,m,T,\eta} \times T \arrow[d, "f"] & \mathcal{M}_{0,m,B,\eta} \times T \arrow[l, "q^+ \times \mathrm{id}_T"'] \arrow[r, "p^+ \times \mathrm{id}_T"] \arrow[d, "f"] & \mathcal{M}_{0,m,G,\eta} \times T \arrow[d, "f"] \\
\mathcal{M}_{0,m,T,\eta} & \mathcal{M}_{0,m,B,\eta} \arrow[l, "q^+"'] \arrow[r, "p^+"] & \mathcal{M}_{0,m,G,\eta}
\end{tikzcd}$$

with f the projection to the first factor. By Lemma 7.4, $\mathcal{A}_{\mu,m,\chi}^+ = \mathcal{A}_\mu \boxtimes \mathcal{F}_\chi$ on $\mathcal{M}_{0,m,G,\eta} \times T$, hence

$$\begin{aligned}
\mathrm{CT}_\eta^+(\mathcal{A}_\mu \boxtimes \mathcal{F}_\chi) &= (q^+ \times \mathrm{id}_T)_! \, (p^+ \times \mathrm{id}_T)^*(\mathcal{A}_\mu \boxtimes \mathcal{F}_\chi) \\
&= (q^+ \times \mathrm{id}_T)_! \big((p^+)^* \mathcal{A}_\mu \boxtimes \mathcal{F}_\chi\big) \\
&= (q^+)_! \, (p^+)^* \mathcal{A}_\mu \ \boxtimes \ \mathcal{F}_\chi \\
&= \mathrm{CT}_\eta(\mathcal{A}_\mu) \ \boxtimes \ \mathcal{F}_\chi,
\end{aligned}$$

where in the third line we used the Künneth formula [LO, Theorem 11.0.14] for the derived lower shriek functor. □

To compute nearby cycles, it is useful to have a trivialization over $\mathrm{Spec}(\mathbb{Z}_p)$. By proper base change for nearby cycles and topological invariance of étale cohomology, we may work with the reduced structure $(\mathcal{M}^t_{0,m,T})_{\mathrm{red}}$ (see the discussion at the start of §9.2); we keep the same notation for complexes on the reduced structure.

Lemma 10.6. *The scheme $(\mathcal{M}_{0,m,T})_{\mathrm{red}}$ is a disjoint union of copies of $\mathrm{Spec}(\mathbb{Z}_p)$ indexed by $\nu \in X_*(T)$ with $\nu \le m$. Over each such copy, $(\mathcal{M}^t_{0,m,T})_{\mathrm{red}}$ identifies with $\mathrm{Spec}(\mathbb{Z}_p) \times T$. Moreover, this identification can be made via a canonical section can_2.*

Proof. For each ν with $m^- \le \nu_i \le m^+$, set $\mathcal{U}_i := t^{\nu_i}\mathbb{Z}_p[t]$. The tuple $(\mathcal{U}_1, \dots, \mathcal{U}_n)$ gives a $\mathbb{Z}_p$-point of $(\mathcal{M}_{0,m,T})_{\mathrm{red}}$ and thus a closed embedding $\mathrm{Spec}(\mathbb{Z}_p) \xrightarrow{\nu} (\mathcal{M}_{0,m,T})_{\mathrm{red}}$ (a section of a separated morphism). Over a field, we used previously (when normalizing CT_η) that $(\mathrm{Gr}_T)_{\mathrm{red}}$ is a disjoint union of points indexed by arbitrary $\nu \in X_*(T)$, so the embeddings $\mathrm{Spec}(\mathbb{Z}_p) \xrightarrow{\nu} (\mathcal{M}_{0,m,T})_{\mathrm{red}}$ cover $(\mathcal{M}_{0,m,T})_{\mathrm{red}}$ as a disjoint union. Pulling back to $\mathcal{M}^t_{0,m,T}$, we obtain a disjoint union of T-torsors over $\mathrm{Spec}(\mathbb{Z}_p)$, all of which are trivial. Thus $(\mathcal{M}^t_{0,m,T})_{\mathrm{red}} \cong \bigsqcup_{\nu \le m} \big(\mathrm{Spec}(\mathbb{Z}_p) \times T\big)$. For later computations, we choose canonical sections can_2 on each $\mathrm{Spec}(\mathbb{Z}_p)$, which amounts to choosing trivializations of the determinant line bundles attached to $(\mathcal{U}_1, \dots, \mathcal{U}_n)$; these are provided by [HLS, Lemma 7.2.1] using the basis e_i of $\mathbb{V}$, and were used in the proof of Lemma 8.11. □

The trivialization can_2, defined over $\mathrm{Spec}(\mathbb{Z}_p)$, allows an explicit nearby cycle computation. We must compare it on the generic fiber with $\mathrm{can}_{1,m}$.

Lemma 10.7. *For any $\nu \in X_*(T)$ corresponding (as in Lemma 10.6) to a copy of $\mathrm{Spec}(\mathbb{Z}_p) \times T$, the generic fiber trivializations $\mathrm{can}_{1,m}$ and $\mathrm{can}_{2,\eta}$ differ by $w_p^{m^+ - \nu}$ in the sense that*

$$\mathrm{Spec}(\mathbb{Q}_p) \times T \xrightarrow{w_p^{m^+-\nu} \times \mathrm{id}_T} T \times T \xrightarrow{m_T} \mathrm{Spec}(\mathbb{Q}_p) \times T$$

identifies the two trivializations as connected components of $(\mathcal{M}^t_{0,m,T,\eta})_{\mathrm{red}}$. Here the left (resp. right) $\mathrm{Spec}(\mathbb{Q}_p) \times T$ is defined using $\mathrm{can}_{2,\eta}$ (resp. $\mathrm{can}_{1,m}$), $w_p^{m^+-\nu} = (w_p^{m^+-\nu_1}, \cdots, w_p^{m^+-\nu_n}) \in T(\mathbb{Q}_p)$ is viewed as a morphism $\mathrm{Spec}(\mathbb{Q}_p) \to T$, and m_T is the multiplication on T.

Proof. This follows from the constructions: $\mathrm{can}_{1,m}$ sends the distinguished section of the i-th determinant line bundle to $1 \in \mathcal{O}_S$ (Lemma 7.4), while $\mathrm{can}_{2,\eta}$ sends it to $w_p^{m^+-\nu} \in \mathcal{O}_S$ ([HLS, Lemma 7.2.1]). □

Lemma 10.8. *For $\nu \in X_*(T)$ as in Lemma 10.7, the restriction of $\mathrm{CT}^t_\eta(\mathcal{A}^+_{\mu,m,\chi})$ to the generic fiber $\mathrm{Spec}(\mathbb{Q}_p) \times T$ of the reduced component indexed by ν is*

$$\big(K_{\nu,m,\chi} \otimes \mathrm{CT}_{\eta,\nu}(\mathcal{A}_\mu)\big) \ \boxtimes\ \mathcal{F}_\chi,$$

where $K_{\nu,m,\chi}$ is the rank-1 local system on $\mathrm{Spec}(\mathbb{Q}_p)$ obtained by pulling back $\mathcal{F}_\chi$ along $\mathrm{Spec}(\mathbb{Q}_p) \hookrightarrow T$ defined by $w_p^{m^+-\nu} \in T(\mathbb{Q}_p)$ (as in Lemma 10.7) and $\mathrm{CT}_{\eta,\nu}(\mathcal{A}_\mu)$ is the restriction of $\mathrm{CT}_\eta(\mathcal{A}_\mu)$ to the reduced component of $\mathcal{M}_{0,m,\eta}$ indexed by ν.

Proof. By Lemma 10.7, we must compute $(w_p^{\nu-m^+} \times \mathrm{id}_T)^*\, m_T^*\big(\mathrm{CT}_{\eta,\nu}(\mathcal{A}_\mu) \boxtimes \mathcal{F}_\chi\big)$; the claim then follows from Lemma 8.7. □

We can now justify the following identification announced above.

Lemma 10.9. *The semisimple trace of Frobenius function (in the sense of Lemma 10.3) attached to $\mathrm{CT}^t_{\bar{s}}\big(R\Psi_{\mathcal{M}^t_{0,m,G}}(\mathcal{A}^+_{\mu,m,\chi})\big)$ is precisely $c_T^G(\tau^{ss}_{\mu,m,\chi})$.*

Proof. Let V_μ be the μ-highest weight irreducible algebraic representation of ${}^LG$. By [HR21, Corollary 3.12], under geometric Satake, $\omega_G(\mathcal{A}_\mu) = V_\mu$ (with ω_G as in Lemma 10.2). Write $\overline{\mathbb{Q}}_{\ell,\nu}$ for the rank-1 constant sheaf supported on the ν-component of $\mathcal{M}_{0,m,T,\eta}$. Since $\omega_T(\overline{\mathbb{Q}}_{\ell,\nu}) = \nu \in X_*(T) =$

$X^*(\widehat{T})$, Lemmas 10.4 and 10.2 imply that $\mathrm{CT}^t_{\bar{s}}\big(R\Psi_{\mathcal{M}^t_{0,m,G}}(\mathcal{A}^+_{\mu,m,\chi})\big)$ is supported on those ν appearing in $V_\mu|_{{}^LT}$. By [HR21, Lemma 7.10], for such ν we have

$$\langle 2\rho_B, \nu\rangle \equiv d_\mu \pmod 2.$$

and $d_W = 0$ there since Schubert strata in Gr_T have dimension 0. Hence, by Lemma 10.3, the function attached to $\mathrm{CT}^t_{\bar{s}}\big(R\Psi_{\mathcal{M}^t_{0,m,G}}(\mathcal{A}^+_{\mu,m,\chi})\big)$ equals ${}^p c^G_T\big((-1)^{d_\mu}\tau^{ss}_{\mu,m,\chi}\big) = c^G_T(\tau^{ss}_{\mu,m,\chi})$, as the sign $(-1)^{\langle 2\rho_B,\nu\rangle}$ is canceled by $(-1)^{d_\mu}$ on the support. □

We can now prove part of our main theorem.

Theorem 10.10. *If m^+ is divisible by $p-1$, then $\tau^{ss}_{\mu,m,\chi} = z^{ss}_{\mu,\chi}$.*

Proof. Since c^G_T is injective, it suffices to show $c^G_T(\tau^{ss}_{\mu,m,\chi}) = c^G_T(z^{ss}_{\mu,\chi})$. View both sides as functions on $\mathrm{Fl}^t_T(\mathbb{F}_p)$. For any $\nu \in X_*(T)$ corresponding to a copy $\mathrm{Spec}(\mathbb{Z}_p)\times T$ as in Lemma 10.6, we compare the two functions on $\mathrm{Spec}(\mathbb{F}_p)\times T \xrightarrow{\nu} \mathcal{M}^t_{0,m,T,s} \hookrightarrow \mathrm{Fl}^t_T$.

First, $c^G_T(\tau^{ss}_{\mu,m,\chi})$ is, by Lemma 10.9, the semisimple trace function attached to

$$R\Psi_{\mathcal{M}^t_{0,m,T}}\big(\mathrm{CT}^t_\eta(\mathcal{A}^+_{\mu,m,\chi})\big).$$

On the generic fiber of the reduced ν-component, Lemma 10.8 gives

$$\mathrm{CT}^t_\eta(\mathcal{A}^+_{\mu,m,\chi}) = (K_{\nu,m,\chi}\otimes \mathrm{CT}_{\eta,\nu}(\mathcal{A}_\mu))\boxtimes \mathcal{F}_\chi.$$

Let $m(\mu,\nu)$ be the multiplicity of ν in $V_\mu|_{\widehat{T}}$. By Lemma 10.2, $\mathrm{CT}_{\eta,\nu}(\mathcal{A}_\mu) = \overline{\mathbb{Q}}_\ell^{m(\mu,\nu)}$, so

$$\mathrm{CT}^t_\eta(\mathcal{A}^+_{\mu,m,\chi}) = K^{m(\mu,\nu)}_{\nu,m,\chi}\boxtimes \mathcal{F}_\chi.$$

Since nearby cycles commute with external products ([Il]) and $R\Psi_T(\mathcal{F}_\chi) = \mathcal{F}_\chi$ ([HLS, Lemma 16.1.1]), we get

$$R\Psi_{\mathcal{M}^t_{0,m,T}}\big(\mathrm{CT}^t_\eta(\mathcal{A}^+_{\mu,m,\chi})\big) \;=\; R\Psi_{\mathrm{Spec}(\mathbb{Z}_p)}(K_{\nu,m,\chi})^{m(\mu,\nu)} \;\boxtimes\; \mathcal{F}_\chi.$$

By definition of $K_{\nu,m,\chi}$, the semisimple trace of $R\Psi_{\mathrm{Spec}(\mathbb{Z}_p)}(K_{\nu,m,\chi})$ is

$$\mathrm{Tr}^{ss}\big(\Phi \mid R\Psi_{\mathrm{Spec}(\mathbb{Z}_p)}(K_{\nu,m,\chi})_{\bar{s}}\big) = \begin{cases} 1, & \chi\circ(\nu-m^+)\big(\mathbb{F}_p^\times\big) = \chi\circ\nu\big(\mathbb{F}_p^\times\big) = 1,\\ 0, & \text{otherwise.}\end{cases}$$

Since the function attached to $\mathcal{F}_\chi$ is χ^{-1} ([HLS, Lemma 15.4.1]), the resulting function is $m(\mu,\nu)$ times the function in Lemma 9.5 (with μ replaced by ν).

Second, $c^G_T(z^{ss}_{\mu,\chi})$ is the function for T induced by the distribution $Z_{V_\mu|_{\widehat{T}}}$ (see (9.2)), which follows from the description of c^G_T on the Bernstein side [Hai12, Sec. 5.1] and the compatibility of $\pi\mapsto\varphi_\pi$ with parabolic induction [FS, Th. I.9.6]. Using this and Lemma 9.5, on the ν-component $c^G_T(z^{ss}_{\mu,\chi})$ equals $m(\mu,\nu)$ times the same function. Hence the two functions agree. □

Recall that $\tau^{ss}_{\mu,m}$ is $(-1)^{d_\mu}$ times the semisimple trace function attached to $R\Psi_{\mathcal{M}^t_{0,m,G}}(\mathcal{A}^+_\mu)$. The centrality of $\tau^{ss}_{\mu,m}$ was conjectured in [HLS]; it now follows directly from Theorem 10.10 at least when m^+ is divisible by $p-1$.

Corollary 10.11. *If m^+ is divisible by $p-1$, then $\tau^{ss}_{\mu,m}\in\mathcal{H}(G,I^+)$ lies in $\mathcal{Z}(G,I^+)$ and $\tau^{ss}_{\mu,m} = |T(\mathbb{F}_p)|z^{ss}_\mu$.*

Proof. We have

$$\tau^{ss}_{\mu,m} \;=\; \sum_\chi \tau^{ss}_{\mu,m,\chi} \;=\; \sum_\chi z^{ss}_{\mu,\chi} \;=\; |T(\mathbb{F}_p)|\cdot z^{ss}_\mu \;\in\; \mathcal{Z}(G,I^+).$$

□

We now generalize Theorem 10.10 and Corollary 10.11 to arbitrary m.

Let $c\in Z(G)(\mathbb{F}_p(\!(t)\!))\subset G(\mathbb{F}_p(\!(t)\!))$ be central. Define a bijection

$$\alpha_c : \mathcal{H}(G(\mathbb{F}_p(\!(t)\!)))\to\mathcal{H}(G(\mathbb{F}_p(\!(t)\!)))$$

by $\alpha_c(f)(z) := f(c^{-1}z)$. One checks easily that $\alpha_c(f)*\alpha_c(g) = \alpha_{c^2}(f*g)$ for all $f, g \in \mathcal{H}(G(\mathbb{F}_p(\!(t)\!)))$, and that α_c preserves $\mathcal{H}(G(\mathbb{F}_p(\!(t)\!)), I^+)$ (indeed, this works at any level). Hence α_c induces a bijection on $\mathcal{Z}(G(\mathbb{F}_p(\!(t)\!)), I^+)$. For $c = \mathrm{diag}(t, \dots, t)$, write α_t. For $n \in \mathbb{Z}$, denote by α_t^n the n-th power of α_t.

Theorem 10.12. *For any m, the function $\tau^{ss}_{\mu,m} \in \mathcal{H}(G, I^+)$ lies in the center $\mathcal{Z}(G, I^+)$ and*

$$\tau^{ss}_{\mu,m} \;=\; |T(\mathbb{F}_p)|\alpha_t^{m^+}\big(z^{ss}_{\mu_{(m^+)}}\big), \qquad \text{where } \mu_{(m^+)} := \mu - (m^+, \dots, m^+).$$

Proof. It suffices to prove the stated equality. Set $m' = (m'^+, m'^-) := (0, m^- - m^+)$. There is a natural isomorphism

$$t_{m,m'}:\; \mathcal{M}_{0,m,G} \xrightarrow{\;\sim\;} \mathcal{M}_{0,m',G}, \qquad \mathcal{W}_\bullet \longmapsto \mathcal{W}'_\bullet := t^{-m^+}\mathcal{W}_\bullet.$$

This lifts to isomorphisms $\mathcal{M}^t_{0,m,G} \cong \mathcal{M}^t_{0,m',G}$ and $\mathcal{M}^t_{1,m,G} \cong \mathcal{M}^t_{1,m',G}$ (denoted again $t_{m,m'}$). The map $t_{m,m'}$ sends $I_{m,\eta}$-orbits on $\mathcal{M}_{0,m,G}$ to $I_{m',\eta}$-orbits on $\mathcal{M}_{0,m',G}$. Using the embedding in Lemma 6.2, we obtain

$$\alpha_t^{-m^+}(\tau^{ss}_{\mu,m}) \;=\; (t_{m,m'})_*(\tau^{ss}_{\mu,m}) \;=\; \tau^{ss}_{\mu_{(m^+)},m'} \;=\; |T(\mathbb{F}_p)| z^{ss}_{\mu_{(m^+)}},$$

with $\mu_{(m^+)} = \mu - (m^+, \dots, m^+)$. This implies the claim. $\square$

Remark 10.13. A consequence of Theorem 10.12 is $z^{ss}_\mu = \alpha_t^{p-1}(z^{ss}_{\mu_{(p-1)}})$ for $\mu_{(p-1)} := \mu - (p-1, \dots, p-1)$. Moreover, when $\chi = \mathrm{triv}$ is the trivial character, we have $z^{ss}_{\mu,\mathrm{triv}} = \alpha_t\big(z^{ss}_{\mu_{(1)},\mathrm{triv}}\big)$ with $\mu_{(1)} := \mu - (1, \dots, 1)$; thus in the iwhaori case, α_t produces no new central function.

11. Extension to $\mathbb{F}_q$

So far we have been working with $\mathbb{F}_p$-points. Let $r \geq 1$ be a positive integer and $q := p^r$. Denote by $\mathbb{Q}_q$ the unramified extension of $\mathbb{Q}_p$ of degree r. Let $\mathbb{Z}_q$ be the ring of integers of $\mathbb{Q}_q$ and $\mathbb{F}_q$ the residue field of $\mathbb{Q}_q$. In this section, we denote the set of characters with values in $\overline{\mathbb{Q}}_\ell^\times$ of $T(\mathbb{F}_q)$ (resp. $T(\mathbb{F}_p)$) by $T(\mathbb{F}_q)^*$ (resp. $T(\mathbb{F}_p)^\vee$). By $I_r \subset G(\mathbb{F}_q(\!(t)\!))$ (resp. $I_r^+ \subset G(\mathbb{F}_q(\!(t)\!))$) we mean the Iwahori subgroup of $G(\mathbb{F}_q(\!(t)\!))$ (resp. its pro-unipotent radical). We generalize results from previous sections to $\mathbb{F}_q$-points. For more details, we refer to [HLS, Sec. 16].

Recall that in §9.3 we constructed z^{ss}_μ (and, for $\chi \in T(\mathbb{F}_p)^\vee$, the elements $z^{ss}_{\mu,\chi}$) for the local field $\mathbb{F}_p(\!(t)\!)$. Replacing $\mathbb{F}_p(\!(t)\!)$ with $\mathbb{F}_q(\!(t)\!)$, one similarly obtains $z^{ss}_{\mu,r} \in \mathcal{Z}\big(G(\mathbb{F}_q(\!(t)\!)), I_r^+\big)$ and, for $\chi' \in T(\mathbb{F}_q)^\vee$, the elements $z^{ss}_{\mu,r,\chi'} \in \mathcal{Z}\big(G(\mathbb{F}_q(\!(t)\!)), I_r, \chi'\big)$.

On the other hand, there are two natural ways to generalize $\tau^{ss}_{\mu,m}$ to $\mathbb{F}_q$-points. First, define

$$\tau^{ss}_{\mu,m,r,\chi} : \mathcal{M}^t_{0,m,G}(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell, \quad x \longmapsto (-1)^{d_\mu}\,\mathrm{Tr}^{ss}\Big(\Phi^r \mid (R\Psi_{\mathcal{M}^t_{0,m,G}}(\mathcal{A}^+_{\mu,m,\chi})_{\bar{x}})\Big),$$

where we use the same symbols as in §9.2. Similarly, define $\tau^{ss}_{\mu,m,r}$ by replacing $\mathcal{A}^+_{\mu,m,\chi}$ with $\mathcal{A}^+_{\mu,m}$ in the formula above, and note that $\tau^{ss}_{\mu,m,r} = \sum_{\chi \in T(\mathbb{F}_p)^\vee} \tau^{ss}_{\mu,m,r,\chi}$. Viewed as an element of $\mathcal{H}(G(\mathbb{F}_q(\!(t)\!)), I_r^+)$, we have $\tau^{ss}_{\mu,m,r,\chi} \in \mathcal{H}(G(\mathbb{F}_q(\!(t)\!)), I_r, \chi')$ where $\chi' = \chi \circ N_{\mathbb{F}_q/\mathbb{F}_p}$ and $N_{\mathbb{F}_q/\mathbb{F}_p} : \mathbb{F}_q^\times \to \mathbb{F}_p^\times$ is the norm map. This implies that when χ' does not factor through the norm map, the χ'-component (in the sense of Lemma 9.2 as we will show that $\tau^{ss}_{\mu,m,r}$ is central) of $\tau^{ss}_{\mu,m,r}$ is zero.

Second, recall that the ramified $T(\mathbb{F}_p)$-cover $\pi : \mathcal{M}^t_{1,m,G} \to \mathcal{M}^t_{0,m,G}$ was defined as the space of $(p-1)$-st roots of the sections $\varphi_i(\alpha_i)$ (Definition 8.1), and that $\mathcal{A}^+_{\mu,\chi}$ is the χ-monodromic component of $\mathcal{A}^+_\mu$ (Lemma 8.10). We now generalize these constructions to q. Define a $T(\mathbb{F}_q)$-ramified cover

$$\pi^r:\; \mathcal{M}^{t,r}_{1,m,G} \longrightarrow \mathcal{M}^t_{0,m,G}$$

as the space of $(q-1)$-st roots of the sections $\varphi_i(\alpha_i)$. When $r = 1$, π^r is just π. As before, obtain a sheaf $\mathcal{A}^{+,r}_{\mu,m} := \pi^r_*\pi^{r*}(\mathcal{B}^+_\mu)$ on $\mathcal{M}^+_{0,m,\mathbb{Q}_p}$. After base change, we get a decomposition on $\mathcal{M}^+_{0,m,\mathbb{Q}_q}$:

$$\mathcal{A}^{+,r}_{\mu,m} \;=\; \bigoplus_{\chi' \in T(\mathbb{F}_q)^*} \mathcal{A}^{+,r}_{\mu,m,\chi'},$$

where each $\mathcal{A}^{+,r}_{\mu,m,\chi'}$ is χ'-monodromic. Define

$$\tau^{r,ss}_{\mu,m,\chi'} : \mathcal{M}^t_{0,m,G}(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell, \quad x \longmapsto (-1)^{d_\mu} \operatorname{Tr}^{ss}\Big(\Phi^r \mid (R\Psi_{\mathcal{M}^t_{0,m,G}}(\mathcal{A}^{+,r}_{\mu,m,\chi'})_{\bar{x}})\Big).$$

Similarly, define $\tau^{r,ss}_{\mu,m}$ and note that $\tau^{r,ss}_{\mu,m} = \sum_{\chi' \in T(\mathbb{F}_q)^\vee} \tau^{r,ss}_{\mu,m,\chi'}$.

The following proposition relates $\tau^{ss}_{\mu,m,r,\chi}$ and $\tau^{r,ss}_{\mu,m,\chi'}$.

Proposition 11.1. *For any $\chi \in T(\mathbb{F}_p)^\vee$ and $\chi' = \chi \circ N_{\mathbb{F}_q/\mathbb{F}_p}$, we have $\tau^{ss}_{\mu,m,r,\chi} = \tau^{r,ss}_{\mu,m,\chi'}$.*

Proof. This is part of [HLS, Proposition 16.1]. □

Theorem 10.12 generalizes to $\mathbb{F}_q$-points.

Theorem 11.2. *For any m, the function $\tau^{r,ss}_{\mu,m} \in \mathcal{H}(G(\mathbb{F}_q(\!(t)\!)), I_r^+)$ lies in the center and*

$$\tau^{r,ss}_{\mu,m} \;=\; |T(\mathbb{F}_q)|\alpha_t^{m^+}\big(z^{ss}_{\mu_{(m^+)},r}\big), \qquad \text{where } \mu_{(m^+)} := \mu - (m^+, \dots, m^+).$$

Proof. Repeat the arguments from the previous sections with p replaced by q. □

Corollary 11.3. *For any m, the function $\tau^{ss}_{\mu,m,r} \in \mathcal{H}(G(\mathbb{F}_q(\!(t)\!)), I_r^+)$ lies in the center.*

Proof. Observing that χ' factors through the norm map if and only if any element in its Weyl group orbit factors through the norm map, the result follows from Proposition 11.1, Theorem 11.2, and Lemma 9.2. □

12. The case $G = \mathrm{GSp}_{2g}$

In this section, we assume $G = \mathrm{GSp}_{2g}$. Recall that the general symplectic group is defined by the symplectic form $\langle \cdot, \cdot \rangle$ on $\mathbb{Z}^{2g}$ with matrix

$$\begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}$$

where J is the antidiagonal identity matrix. The proofs of the results in this section are essentially the same as in the case $G = \mathrm{GL}_n$; rather than repeating everything, we focus only on the differences. We retain the notation from previous sections. In particular, here $G = \mathrm{GSp}_{2g}$, T is its diagonal torus, and B is the standard "upper" Borel subgroup. Of course, the rank of the ambient space $\mathbb{V}$ is assumed to be $2g$ in this section.

Recall the definition of $\mathcal{M}_{0,m,G}$ for GSp_{2g} ([HN, Definition 7] and [HLS, Definition 7.1.1]).

Definition 12.1. Let $\mathcal{M}_{0,m,G}$ denote the moduli space that associates to any scheme S over $\mathrm{Spec}(\mathbb{Z}_p)$ the set of chains $(\mathcal{W}_0 \subset \mathcal{W}_1 \subset \cdots \subset \mathcal{W}_g)$ of $\mathcal{O}_S[t]$-submodules of $\mathcal{O}_S[t, t^{-1}, (t + w_p)^{-1}]^{2g}$ fitting into a commutative diagram with injective morphisms

$$\begin{tikzcd}
t^{m^-}\mathbb{V}_0[t]_{\mathcal{O}_S} \arrow[r] & t^{m^-}\mathbb{V}_1[t]_{\mathcal{O}_S} \arrow[r] & \cdots \arrow[r] & t^{m^-}\mathbb{V}_g[t]_{\mathcal{O}_S} \\
\mathcal{W}_0 \arrow[r] \arrow[u] & \mathcal{W}_1 \arrow[r] \arrow[u] & \cdots \arrow[r] & \mathcal{W}_g \arrow[u] \\
t^{m^+}\mathbb{V}_0[t]_{\mathcal{O}_S} \arrow[r] \arrow[u] & t^{m^+}\mathbb{V}_1[t]_{\mathcal{O}_S} \arrow[r] \arrow[u] & \cdots \arrow[r] & t^{m^+}\mathbb{V}_g[t]_{\mathcal{O}_S} \arrow[u]
\end{tikzcd}$$

where

- $\mathcal{W}_i / t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S} \subset t^{m^-}\mathbb{V}_i[t]_{\mathcal{O}_S} / t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S}$ is locally a direct factor of rank $g(m^+ - m^-)$ as an $\mathcal{O}_S$-module, and
- $\mathcal{W}_0$ is self-dual with respect to the pairing $t^{-m^+ - m^-}\langle \cdot, \cdot \rangle$ and $\mathcal{W}_g$ is self-dual with respect to the pairing $t^{-m^+ - m^-}(t + w_p)\langle \cdot, \cdot \rangle$.

For $0 \le i \le g$, define the dual with respect to $t^{-m^+-m^-}\langle\cdot,\cdot\rangle$ (similarly for $t^{-m^+-m^-}(t+w_p)\langle\cdot,\cdot\rangle$) by

$$\mathcal{W}_i^\perp := \{v \in \mathcal{O}_S[t, t^{-1}, (t+w_p)^{-1}]^{2g} \mid t^{-m^+-m^-}\langle v, w\rangle \in \mathcal{O}_S[t],\ \forall\, w \in \mathcal{W}_i\}.$$

For $g \le i \le 2g$, set $\mathcal{W}_i = (t+w_p)^{-1}\mathcal{W}_{2g-i}^\perp$. Note that $\mathcal{W}_g = (t+w_p)^{-1}\mathcal{W}_g^\perp$ is assumed, so this definition makes sense and yields a complete chain $\mathcal{W}_\bullet = (\mathcal{W}_0 \subset \cdots \subset \mathcal{W}_{2g} = (t+w_p)^{-1}\mathcal{W}_0)$. The following lemma is used implicitly in [HLS].

Lemma 12.2. *The construction of complete chains $\mathcal{W}_\bullet = (\mathcal{W}_0 \subset \cdots \subset \mathcal{W}_{2g} = (t+w_p)^{-1}\mathcal{W}_0)$ defines a closed embedding $\mathcal{M}_{0,m,G} \subset \mathcal{M}_{0,m,\mathrm{GL}_{2g}}$.*

Proof. Since $\langle\cdot,\cdot\rangle$ is a perfect pairing, for $g \le i \le 2g$ the quotient

$$\mathcal{W}_i / t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S} \subset t^{m^-}\mathbb{V}_i[t]_{\mathcal{O}_S} / t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S}$$

is locally a direct factor of rank $\mathrm{rk}_R(t^{m^-}\mathbb{V}_{2g-i}[t]_{\mathcal{O}_S}/\mathcal{W}_{2g-i}) = g(m^+ - m^-)$ as an $\mathcal{O}_S$-module, so the embedding is well-defined. The additional requirement that $\mathcal{W}_0$ and $\mathcal{W}_g$ are self-dual is a closed condition. $\square$

Remark 12.3. Once a $\mathbb{G}_m$-action is defined on $\mathcal{M}_{0,m,G}$ similarly to before, Lemma 5.6 together with Lemma 12.2 implies that the $\mathbb{G}_m$-action on $\mathcal{M}_{0,m,G}$ is Zariski locally linearizable, since the embedding $\mathcal{M}_{0,m,G} \subset \mathcal{M}_{0,m,\mathrm{GL}_{2g}}$ is $\mathbb{G}_m$-equivariant. Hence hyperbolic localization produces representable objects in this section.

Let $2m := (2m^+, 2m^-)$. To define the multiplication map $\mathrm{mult} : \mathcal{M}_{0,m,\mathbb{G}_m} \times \mathcal{M}_{0,m,\mathbb{G}_m} \to \mathcal{M}_{0,2m,\mathbb{G}_m}$ (induced from multiplication on $\mathbb{G}_m$), we use the following equivalent definition of $\mathcal{M}_{0,m,\mathbb{G}_m}$.

Lemma 12.4. *The moduli space $\mathcal{M}_{0,m,\mathbb{G}_m}$ associates to any scheme S over $\mathrm{Spec}(\mathbb{Z}_p)$ the set of $\mathcal{O}_S$-lattices $\mathcal{U}_0 \subset \mathcal{O}_S(\!(t)\!)$ such that $t^{m^+}\mathcal{O}_S[\![t]\!] \subset \mathcal{U}_0 \subset t^{m^-}\mathcal{O}_S[\![t]\!]$.*

Proof. This is the equivalent definition (i) in Definition 3.1. $\square$

Definition 12.5. The multiplication map $\mathrm{mult} : \mathcal{M}_{0,m,\mathbb{G}_m} \times \mathcal{M}_{0,m,\mathbb{G}_m} \to \mathcal{M}_{0,2m,\mathbb{G}_m}$ is defined as follows. For any scheme S over $\mathrm{Spec}(\mathbb{Z}_p)$ and any $\mathcal{U}_0, \mathcal{U}_0' \in \mathcal{M}_{0,m,\mathbb{G}_m}$ as in Lemma 12.4, set

$$\mathrm{mult}(\mathcal{U}_0, \mathcal{U}_0') := \mathcal{U}_0 \otimes_{\mathcal{O}_S[\![t]\!]} \mathcal{U}_0'.$$

This is well-defined since $t^{2m^+}\mathcal{O}_S[\![t]\!] \subset \mathcal{U}_0 \otimes_{\mathcal{O}_S[\![t]\!]} \mathcal{U}_0' \subset t^{2m^-}\mathcal{O}_S[\![t]\!]$ and $\mathcal{U}_0 \otimes_{\mathcal{O}_S[\![t]\!]} \mathcal{U}_0'$ is $\mathcal{O}_S[\![t]\!]$-projective.

Using this multiplication map, we define $\mathcal{M}_{0,m,T}$.

Definition 12.6. Let $\mathcal{M}_{0,m,T}$ be the closed subscheme of $\mathcal{M}_{0,m,\mathbb{G}_m^{2g}} = (\mathcal{M}_{0,m,\mathbb{G}_m})^{2g}$ that associates to any scheme S over $\mathrm{Spec}(\mathbb{Z}_p)$ the set of $2g$-tuples $(\mathcal{U}_1, \ldots, \mathcal{U}_{2g})$ such that $\mathrm{mult}(\mathcal{U}_i, \mathcal{U}_{2g+1-i}) = t^{m^++m^-}\mathcal{O}_S[\![t]\!] \subset \mathcal{O}_S(\!(t)\!)$, where mult is defined in Definition 12.5.

Remark 12.7. Although the maximal split torus $T \subset G$ is isomorphic to $\mathbb{G}_m^{g+1}$, the scheme $\mathcal{M}_{0,m,T}$ is **not** defined to be isomorphic to $\mathcal{M}_{0,m,\mathbb{G}_m^{g+1}}$. In fact, $\mathcal{M}_{0,m,T} \cong \mathcal{M}_{0,m,\mathbb{G}_m^g}$, since the first g entries determine the whole $2g$-tuple; this corresponds to the diagonal torus of Sp_{2g}.

Lemma 12.8. *The restriction of the embedding $\mathcal{M}_{0,m,\mathbb{G}_m^{2g}} \subset \mathcal{M}_{0,m,\mathrm{GL}_{2g}}$ (Definition 4.8) to $\mathcal{M}_{0,m,T}$ factors through $\mathcal{M}_{0,m,G} \subset \mathcal{M}_{0,m,\mathrm{GL}_{2g}}$ (Lemma 12.2), and the resulting diagram*

$$\begin{tikzcd}
\mathcal{M}_{0,m,T} \arrow[r] \arrow[d] & \mathcal{M}_{0,m,\mathbb{G}_m^{2g}} \arrow[d] \\
\mathcal{M}_{0,m,G} \arrow[r] & \mathcal{M}_{0,m,\mathrm{GL}_{2g}}
\end{tikzcd}$$

is Cartesian.

Proof. This follows from the fact that the restriction of the symplectic form $\langle\cdot,\cdot\rangle$ to the $(i, 2g+1-i)$ components of $\mathbb{V}$ corresponds to multiplication of elements. $\square$

As in Lemma 3.3 (resp. Lemma 3.4), the R-valued points of the ind-scheme Gr_G (resp. Fl_G) can be described by lattices $\mathfrak{L}$ (resp. t-periodic complete lattice chains $\mathfrak{L}_\bullet$) in $R(\!(t)\!)^{2g}$ such that $\mathfrak{L}^{\perp_c} = \mathfrak{L}$ (resp. $\mathfrak{L}_i^{\perp_c} = \mathfrak{L}_{-i}$ for all $i \in \mathbb{Z}$) for some $c \in R(\!(t)\!)^\times$, where $\mathfrak{L}^{\perp_c}$ denotes the dual with respect to $c^{-1}\langle\cdot,\cdot\rangle$:

$$\mathfrak{L}^{\perp_c} := \{x \in R(\!(t)\!)^{2g} \mid c^{-1}\langle x, v\rangle \in R[\![t]\!] \ \forall\, v \in \mathfrak{L}\}.$$

Given a lattice $\mathfrak{L}$ (resp. a t-periodic complete lattice chain $\mathfrak{L}_\bullet$) in $R(\!(t)\!)^{2g}$, if $\mathfrak{L} = \mathfrak{L}^{\perp_{c_1}} = \mathfrak{L}^{\perp_{c_2}}$ (resp. $\mathfrak{L}_i^{\perp_{c_1}} = \mathfrak{L}_i^{\perp_{c_2}} = \mathfrak{L}_{-i}$ for all i) for some $c_1, c_2 \in R(\!(t)\!)^\times$, then $\frac{c_1}{c_2} \in R[\![t]\!]^\times$. Thus we obtain the similitude map $\mathrm{Gr}_G \to \mathrm{Gr}_{\mathbb{G}_m}$ (resp. $\mathrm{Fl}_G \to \mathrm{Gr}_{\mathbb{G}_m}$) by $\mathfrak{L} \mapsto c_1$ (resp. $\mathfrak{L}_\bullet \mapsto c_1$). Forgetting self-duality gives natural embeddings $\mathrm{Gr}_G \subset \mathrm{Gr}_{\mathrm{GL}_{2g}}$ and $\mathrm{Fl}_G \subset \mathrm{Fl}_{\mathrm{GL}_{2g}}$. As in Lemma 4.4 (resp. Lemma 4.6), there is a natural closed embedding $\mathcal{M}_{0,m,G,\eta} \subset \mathrm{Gr}_G$ (resp. $\mathcal{M}_{0,m,G,s} \subset \mathrm{Fl}_G$).

Remark 12.9. Remark 12.7 and Lemma 12.8 may look surprising at first glance, but this is tied to the choice of the pairing $t^{-m^+-m^-}\langle\cdot,\cdot\rangle$ in Definition 12.1, which makes $\mathcal{M}_{0,m,G}$ more "reduced" in the following sense. For simplicity, assume $m^+ = -m^-$ and work over the generic fiber. Denote by $f : \mathrm{Gr}_G \to \mathrm{Gr}_{\mathbb{G}_m}$ the similitude map. Then the restriction $f|_{\mathcal{M}_{0,m,G,\eta}}$ factors through $(\mathrm{Gr}_{\mathbb{G}_m})_{\mathrm{red}} \subset \mathrm{Gr}_{\mathbb{G}_m}$. Concretely, writing $\mathrm{Gr}_{\mathbb{G}_m}$ as a disjoint union of infinitesimal points indexed by $\mathbb{Z} = X_*(\mathbb{G}_m)$, the image of $f|_{\mathcal{M}_{0,m,G,\eta}}$ lies on the point pt_0 indexed by $0 \in \mathbb{Z}$ and in fact factors through the reduced point $\mathrm{Spec}(\mathbb{Q}_p) = (pt_0)_{\mathrm{red}} \subset pt$. In this case, the image of the embedding $\mathcal{M}_{0,m,G,\eta} \subset \mathrm{Gr}_G$ is exactly the closed subscheme of $\mathrm{Gr}_{\mathrm{Sp}_{2g}} \subset \mathrm{Gr}_G$ bounded by m. For general m, the image of $f|_{\mathcal{M}_{0,m,G,\eta}}$ lies in the reduced point indexed by $m^+ + m^-$.

Although $\mathrm{GL}_2 = \mathrm{GSp}_2$, the scheme $\mathcal{M}_{0,m,\mathrm{GSp}_2}$ is defined differently from $\mathcal{M}_{0,m,\mathrm{GL}_2}$ in that:

- $\mathcal{M}_{0,m,\mathrm{GSp}_2}$ lies in a single connected component indexed by $m^+ + m^- \in \mathbb{Z}$; and
- $\mathcal{M}_{0,m,\mathrm{GSp}_2}$ is more "reduced" in the sense above.

Definition 12.10. Let $\mathcal{M}_{0,m,B}$ denote the moduli space that associates to any scheme S over $\mathrm{Spec}(\mathbb{Z}_p)$ the $\mathcal{O}_S[\![t]\!]$-lattices $t^{m^+}\mathbb{V}_0[t]_{\mathcal{O}_S} \subset \mathcal{W}_0 \subset t^{m^-}\mathbb{V}_0[t]_{\mathcal{O}_S}$ such that

- $\mathcal{W}_0/t^{m^+}\mathbb{V}_0[t]_{\mathcal{O}_S} \subset t^{m^-}\mathbb{V}_0[t]_{\mathcal{O}_S}/t^{m^+}\mathbb{V}_0[t]_{\mathcal{O}_S}$ is locally a direct factor of rank $g(m^+ - m^-)$ as an $\mathcal{O}_S$-module; and
- for $\mathcal{W}_{0,i} := \mathcal{W}_0 \cap \mathcal{O}_S(t)^i$ (see Remark 4.11; in particular, $\mathcal{W}_{0,0} = 0$ and $\mathcal{W}_{0,n} = \mathcal{W}_0$), there is an exact sequence $\mathcal{W}_{0,i-1} \to \mathcal{W}_{0,i} \to \mathcal{Q}_i$ where the first arrow is the inclusion and the second is the projection to the i-th component, such that as the image of the projection, the $\mathcal{O}_S[t]$-module $t^{m^+}\mathcal{O}_S[t] \subset \mathcal{Q}_i \subset t^{m^-}\mathcal{O}_S[t]$ defines a point of $\mathcal{M}_{0,m,\mathrm{GL}_1}(S)$ (equivalently, $\mathcal{Q}_i/t^{m^+}\mathcal{O}_S[t] \subset t^{m^-}\mathcal{O}_S[t]/t^{m^+}\mathcal{O}_S[t]$ is locally a direct factor as an $\mathcal{O}_S$-module); and
- $\mathcal{W}_0$ is self-dual with respect to the pairing $t^{-m^+-m^-}\langle\cdot,\cdot\rangle$.

Lemma 12.11. *There is a natural closed embedding $\mathcal{M}_{0,m,B} \subset \mathcal{M}_{0,m,B_{\mathrm{GL}_{2g}}}$, where $B_{\mathrm{GL}_{2g}}$ denotes the upper Borel of GL_{2g}.*

Proof. Compare Definition 4.12 with Definition 12.10. Fixing the rank of $\mathcal{W}_0/t^{m^+}\mathbb{V}_0[t]_{\mathcal{O}_S}$ is an open-and-closed condition. Requiring $\mathcal{W}_0$ to be self-dual is a closed condition. □

Lemma 12.12. *The natural embedding $\mathcal{M}_{0,m,B} \subset \mathcal{M}_{0,m,B_{\mathrm{GL}_{2g}}} \subset \mathcal{M}_{0,m,\mathrm{GL}_{2g}}$ factors through $\mathcal{M}_{0,m,G} \subset \mathcal{M}_{0,m,\mathrm{GL}_{2g}}$, and the diagram*

$$\begin{tikzcd}
\mathcal{M}_{0,m,B} \arrow[r] \arrow[d] & \mathcal{M}_{0,m,B_{\mathrm{GL}_{2g}}} \arrow[d] \\
\mathcal{M}_{0,m,G} \arrow[r] & \mathcal{M}_{0,m,\mathrm{GL}_{2g}}
\end{tikzcd}$$

is Cartesian.

Proof. Recall from Lemma 4.17 that $\mathcal{W}_i := \mathcal{W}_0 + (t + w_p)^{-1}\mathcal{W}_{0,i}$. We must check $\mathcal{W}_i = (t + w_p)^{-1}\mathcal{W}_{2g-i}^{\perp}$ for $g \le i \le 2g$.

Observe that $\mathcal{W}_{0,i}^{\mathrm{orth}} = \mathcal{W}_{0,2g-i}$, where for $\mathcal{W} \subset \mathcal{W}_0$ we set

$$\mathcal{W}^{\mathrm{orth}} := \{\, w \in \mathcal{W}_0 \mid \langle w, v\rangle = 0 \ \forall\, v \in \mathcal{W} \,\}.$$

In particular, $\langle \mathcal{W}_{0,i}, \mathcal{W}_{0,2g-i}\rangle = 0$, which implies

$$\begin{aligned} t^{-m^+-m^-}(t+w_p)\langle \mathcal{W}_i, \mathcal{W}_{2g-i}\rangle &= t^{-m^+-m^-}(t+w_p)\langle \mathcal{W}_0 + (t+w_p)^{-1}\mathcal{W}_{0,i},\ \mathcal{W}_0 + (t+w_p)^{-1}\mathcal{W}_{0,2g-i}\rangle \\ &\subset t^{-m^+-m^-}(t+w_p)\langle \mathcal{W}_0, (t+w_p)^{-1}\mathcal{W}_0\rangle \\ &= \mathcal{O}_S[t]. \end{aligned}$$

Thus $\mathcal{W}_i \subset (t+w_p)^{-1}\mathcal{W}_{2g-i}^{\perp}$. Since $t^{m^+}\mathbb{V}_{2g-i}[t]_{\mathcal{O}_S} \subset \mathcal{W}_{2g-i} \subset t^{m^-}\mathbb{V}_{2g-i}[t]_{\mathcal{O}_S}$, we clearly have $t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S} \subset (t+w_p)^{-1}\mathcal{W}_{2g-i}^{\perp} \subset t^{m^-}\mathbb{V}_i[t]_{\mathcal{O}_S}$. Now $\mathcal{W}_i/t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S} \subset (t+w_p)^{-1}W_{2g-i}^{\perp}/t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S}$ are projective $\mathcal{O}_S$-modules of the same rank, and $(t+w_p)^{-1}\mathcal{W}_{2g-i}^{\perp}/\mathcal{W}_i$ is $\mathcal{O}_S$-projective; hence $\mathcal{W}_i = (t+w_p)^{-1}\mathcal{W}_{2g-i}^{\perp}$. □

We define the morphism $\mathcal{M}_{0,m,B} \to \mathcal{M}_{0,m,T}$ as the morphism induced by the next lemma.

Lemma 12.13. *The restriction of the morphism $\mathcal{M}_{0,m,B_{\mathrm{GL}_{2g}}} \to \mathcal{M}_{0,m,\mathbb{G}_m^{2g}}$ to $\mathcal{M}_{0,m,B} \subset \mathcal{M}_{0,m,B_{\mathrm{GL}_{2g}}}$ factors through $\mathcal{M}_{0,m,T} \subset \mathcal{M}_{0,m,\mathbb{G}_m^{2g}}$.*

Proof. We must check $\mathrm{mult}(\mathcal{Q}_i, \mathcal{Q}_{2g+1-i}) = t^{m^++m^-}\mathcal{O}_S[\![t]\!]$. By Lemma 12.12, $\mathcal{W}_{0,i}^{\mathrm{orth}} = \mathcal{W}_{0,2g-i}$, so the pairing $t^{-m^+-m^-}\langle\cdot,\cdot\rangle$ induces a pairing on $\mathcal{W}_{0,i}/\mathcal{W}_{0,i-1} = \mathcal{Q}_i$ and $\mathcal{W}_{0,2g+1-i}/\mathcal{W}_{0,2g-i} = \mathcal{Q}_{2g+1-i}$ given by $t^{-m^+-m^-}$ times multiplication of elements. Passing to the induced pairing on the quotients

$$\mathcal{Q}_i/t^{m^+}\mathcal{O}_S[t] \subset t^{m^-}\mathcal{O}_S[t]/t^{m^+}\mathcal{O}_S[t] \quad \text{and} \quad \mathcal{Q}_{2g+1-i}/t^{m^+}\mathcal{O}_S[t] \subset t^{m^-}\mathcal{O}_S[t]/t^{m^+}\mathcal{O}_S[t],$$

we obtain

$$(\mathcal{Q}_i/t^{m^+}\mathcal{O}_S[t])^{\mathrm{orth}} = \mathcal{Q}_{2g+1-i}/t^{m^+}\mathcal{O}_S[t],$$

where $^{\mathrm{orth}}$ is defined using the induced pairing on $t^{m^-}\mathcal{O}_S[t]/t^{m^+}\mathcal{O}_S[t]$. This implies

$$\mathrm{mult}(\mathcal{Q}_i, \mathcal{Q}_{2g+1-i}) = t^{m^++m^-}\mathcal{O}_S[\![t]\!].$$

□

To define T-torsors, recall the following lemma.

Lemma 12.14. *For any $\mathcal{W}_\bullet \in \mathcal{M}_{0,m,G}(S)$, there is a canonical isomorphism*

$$\mathrm{Det}\Big[\frac{\mathcal{W}_{i-1}}{t^{m^+}\mathbb{V}_{i-1}[t]} \to \frac{\mathcal{W}_i}{t^{m^+}\mathbb{V}_i[t]}\Big] \otimes_{\mathcal{O}_S} \mathrm{Det}\Big[\frac{\mathcal{W}_{2g-i}}{t^{m^+}\mathbb{V}_{2g-i}[t]} \to \frac{\mathcal{W}_{2g+1-i}}{t^{m^+}\mathbb{V}_{2g+1-i}[t]}\Big] \cong \mathcal{O}_S \tag{12.1}$$

sending the distinguished section $a_i \otimes a_{2g+1-i}$ on the left to the section $w_p^{m^+-m^-}$ on the right. This family of isomorphisms is symmetric in the sense of [HLS, Sec.4.1].

Proof. This is [HLS, Lemma 7.2.3]. □

Definition 12.15. Let $\mathcal{M}_{0,m,G}^t$ be the moduli space over $\mathcal{M}_{0,m,G}$ parametrizing isomorphisms

$$\varphi_i :\ L_i' := \mathrm{Det}\Big[\frac{\mathcal{W}_{i-1}}{t^{m^+}\mathbb{V}_{i-1}[t]_{\mathcal{O}_S}} \to \frac{\mathcal{W}_i}{t^{m^+}\mathbb{V}_i[t]_{\mathcal{O}_S}}\Big] \xrightarrow{\sim} \mathcal{O}_S$$

such that there exists a unit $u \in \mathcal{O}_{\mathcal{M}_{0,m,G}}^\times$ with the property that, for all i, the diagram

$$\begin{tikzcd} L_i' \otimes L_{2g+1-i}' \arrow[r, "\mathrm{can}", "\sim"'] \arrow[d, "\varphi_i\otimes\varphi_{2g+1-i}"', "\wr"] & \mathcal{O}_{\mathcal{M}_{0,m}} \arrow[d, "\wr"', "\times u"] \\ \mathcal{O}_{\mathcal{M}_{0,m}} \otimes \mathcal{O}_{\mathcal{M}_{0,m}} \arrow[r, "\mathrm{mult}", "\sim"'] & \mathcal{O}_{\mathcal{M}_{0,m}} \end{tikzcd} \tag{12.2}$$

commutes, where $\mathrm{mult} : \mathcal{O}_{\mathcal{M}_{0,m}} \otimes \mathcal{O}_{\mathcal{M}_{0,m}} \to \mathcal{O}_{\mathcal{M}_{0,m}}$ is multiplication and can is the symmetric family of Lemma 12.14. Since can is symmetric, $\mathcal{M}_{0,m,G}^t$ is a T-torsor over $\mathcal{M}_{0,m,G}$.

Define $\mathcal{M}_{0,m,T}^t$ and $\mathcal{M}_{0,m,B}^t$ by pulling back the T-torsor $\mathcal{M}_{0,m,G}^t$ along the embeddings $\mathcal{M}_{0,m,T} \subset \mathcal{M}_{0,m,B} \subset \mathcal{M}_{0,m,G}$ from Lemma 4.17, Lemma 12.8, Lemma 12.12 and thus obtaining embeddings $\mathcal{M}_{0,m,T}^t \subset \mathcal{M}_{0,m,B}^t \subset \mathcal{M}_{0,m,G}^t$.

As in the GL_n case (Lemma 7.4), these T-torsors are trivial on the generic fiber.

Lemma 12.16. *The T-torsor $\mathcal{M}^t_{0,m,G,\eta} \to \mathcal{M}_{0,m,G,\eta}$ is trivial. Moreover, there is an $I_{m,\eta}$-equivariant, hence $\mathbb{G}_{m,\eta}$-equivariant, canonical section* $\mathrm{can}_{1,\mathrm{m}}$ *trivializing this T-torsor. The analogous statements hold with G replaced by T or B. For the definition of I_m in this case, see* [HLS, Definition 6.4.1].

Proof. The proof is identical to that of Lemma 7.4. Note that the unit u in Definition 12.15 can be taken to be $w_p^{m^- - m^+}$, which is invertible on the generic fiber. □

With these ingredients in place, all definitions and results from previous sections generalize to the case $G = \mathrm{GSp}_{2g}$.

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